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Cohomology for Generalized Bredon Coefficient Systems and Higher K-Theory

by

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Abstract

Let $\mathcal{C}$ be a generalized based category (see definition 1.2). In this paper, we construct a cohomology theory in the category $\mathcal{B}_R(\mathcal{C})$ of contravariant functors: $\mathcal{C} \to R$-$\mathcal{M}$od where R is a commutative ring with identity, which generalizes Bredon cohomology involving finite, profinite or discrete groups.

We also study higher K-theory of the category $\mathcal{P}(\mathcal{B}_R(\mathcal{C}))$ of finitely generated projective objects in $\mathcal{B}_R(\mathcal{C})$ and the category $\mathcal{M}(\mathcal{B}_R(\mathcal{C}))$ of finitely generated objects in $\mathcal{B}_R(\mathcal{C})$ and obtain some finiteness and other results.

Key Words: generalized Bredon coefficient systems, cohomology, generalized based categories, higher K-theory, finite, profinite and discrete groups.

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Introduction

Let G be a finite group and G-$\mathcal{S}$et^f the category of finite G-sets. A "based category" $\mathcal{C}$ is defined in [10] 9.4 as a category with finite pull backs, finite coproducts and final object satisfying certain axioms (see [10] 9.4.1), including a key property that every object is a finite coproduct of indecomposable objects and the number of isomorphism classes of indecomposable objects is finite. As explained in [10], it is a natural generalization of the category G-$\mathcal{S}$et^f. A Bredon coefficient system is classically defined as a contravariant functor: G-$\mathcal{S}$et$^f \to \mathbb{Z}$-$\mathcal{M}$od. So, if R is a commutative ring with identity, we could also define a Bredon coefficient system for $\mathcal{C}$ as a contravariant functor: $\mathcal{C} \to R$-$\mathcal{M}$od where R-$\mathcal{M}$od is the category of left R-modules, and $\mathcal{C}$ is any based category.

Now let G be any discrete group and G-$\mathcal{S}$et^d, the category of proper discrete G-spaces (see Example 1.5 (ii)). Then G and the objects of G-$\mathcal{S}$et^d are no longer finite. G-$\mathcal{S}$et^d constitutes an example of a new category to be called a "generalized based category" which, we define as a category with coproducts, '$\coprod$', pull backs and final object '$*$' satisfying certain axioms (see Definition 1.2). Note that such a category also has products and initial object 'ϕ' (see [12]).

Now suppose that $\mathcal{C}$ is a generalized based category, and $\overline{\mathcal{C}}$ the full subcategory of indecomposable objects in $\mathcal{C}$ (see Definition 1.1). Let $I_{\overline{\mathcal{C}}}$ denote the set of representatives of isomorphism classes of $\overline{\mathcal{C}}$-objects. Now let $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. We call a contravariant functor $\mathcal{A} \to R\text{-}\mathcal{M}\text{od}$ a generalized Bredon coefficient system for $\mathcal{A}$ and denote by $\mathcal{B}_R(\mathcal{A})$ the category of Bredon coefficient systems $\mathcal{A} \overset{R}{\to} R\text{-}\mathcal{M}\text{od}$.

In this paper, we construct a cohomology theory in $\mathcal{B}_R(\mathcal{C})$ (resp. $\mathcal{B}_R(\overline{\mathcal{C}})$) in a general categorical setting (without mentioning any groups) so that the cohomology theory in $\mathcal{B}_R(\overline{\mathcal{C}})$ extends to that on $\mathcal{B}_R(\mathcal{C})$ as well as generalizes Bredon cohomology theory for $\mathcal{B}_{\mathbb{Z}}(\overline{G\text{-}\mathcal{S}\text{et}^d})$. This paper is the first to construct such cohomology theories in the generality of Bredon (resp. generalized Bredon) coefficient systems in the purely categorical setting of 'based' and 'generalized based' categories. Note that results on generalized based categories apply to based categories but not conversely.

Let $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. We also obtain some results on the Higher K-theory of $\mathcal{P}(\mathcal{B}_R(\mathcal{A}))$ the category of finitely generated projective objects as well as of $\mathcal{M}(\mathcal{B}_R(\mathcal{A}))$ the category of finitely generated objects in $\mathcal{B}_R(\mathcal{A})$. Note that in the terminology of [10] 7.8 or [11], $\overline{\mathcal{C}}$ is an EI category.

We briefly discuss the contents of this paper.

In section 1, we set the stage by defining a generalized based category and providing examples involving finite, profinite, and discrete groups (see Definition 1.2 and Example 1.5).

Next, we construct projective objects in $\mathcal{B}_R(\mathcal{A})$ (see 2.2.4 and 2.2.5) where $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$, and show that given any $M \in \text{ob}(\mathcal{B}_R(\mathcal{A}))$, there exist projective objects $P \in \text{ob}(\mathcal{B}_R(\mathcal{A}))$ and epimorphisms $P \to M$ i.e., $\mathcal{B}_R(\mathcal{A})$ has enough projectives (see theorem 2.3.1). It then follows that every $M \in \text{ob}(\mathcal{B}_R(\mathcal{A}))$ has a projective resolution $P_*(M) \to M$ and so, given any other $N \in \text{ob}(\mathcal{B}_R(\mathcal{A}))$, we obtain a cochain complex $\text{Hom}_{\mathcal{B}_R(\mathcal{A})}(P_*(M), N)$ and hence cohomology groups $H^i(\text{Hom}_{\mathcal{B}_R(\mathcal{A})}(P_*(M), N))$.

In the final section, we study the K-groups $K_n(\mathcal{P}(\mathcal{B}_R(\mathcal{C})))$ and $K_n(\mathcal{M}(\mathcal{B}_R(\mathcal{C})))$ for all $n \geq 0$. Now, recall from [6], [10], 7.6 and [11] that an EI-category is a category in which every endomorphism is an isomorphism. So, it follows from axiom (b) of definition 1.2 that $\overline{\mathcal{C}}$ is an EI-category. Hence the results of [6], [10] 7.6 and [11] on EI-categories hold for $\overline{\mathcal{C}}$. We extend some of these results to $K_n(\mathcal{P}(\mathcal{B}_R(\mathcal{C})))$, $K_n(\mathcal{M}(\mathcal{B}_R(\mathcal{C})))$ while also obtaining new ones. (see 3.1.8, 3.1.9, 3.1.10, 3.2.2).

Notes on Notation

If $\mathcal{A}$ is an exact category in the sense of Quillen [14], we shall write $K_n(\mathcal{A})$ $n \geq 0$ for Quillen's higher K-group $\pi_{n+1}(BQ\mathcal{A})$ (see [14]). If R is a ring with identity, and $\mathcal{P}(R)$ (resp. $\mathcal{M}(R)$) the category of finitely generated projective (resp. finitely generated R-modules) we shall write $K_n(R)$, (resp. $G_n(R)$) for $K_n(\mathcal{P}(R))$ (resp. $K_n(\mathcal{M}(R))$).

If $\mathcal{C}$ is a based (or generalized based) category, (see [10] 9.4.1 and definition 1.2 in this paper) and $\overline{\mathcal{C}}$ the full subcategory of indecomposable objects in $\mathcal{C}$, we shall use 'X' for objects of $\mathcal{C}$ and 'T' for objects of $\overline{\mathcal{C}}$. Thus, by 1.2 (c), any $X \in \mathcal{C}$ has the form $X = \coprod_{j \in J} T_j$ where T_j is in ob($\overline{\mathcal{C}}$), J is some index set and '$\coprod$' denotes the coproduct or categorical sum in $\mathcal{C}$. If $\mathcal{C}$ is a based category, then J is a finite index set. We shall write $\mathbb{Z}$ for the ring of rational integers and $\mathbb{Z}$-$\mathcal{M}$od for the category of Abelian groups.

1. Generalized Based Categories and some examples

Definition 1.1 Let $\mathcal{C}$ be a category with coproducts '$\coprod$' and initial object ϕ. A $\mathcal{C}$-object X is said to be indecomposable if $X \simeq X_1 \coprod X_2$ implies either $X_1 \simeq \phi$ or $X_2 \simeq \phi$.

Define a pre-ordering $<$ on $\mathcal{C}$ by $X < Y$ if $\mathrm{Hom}_{\mathcal{C}}(X,Y) \neq \phi$, and an equivalence relation $\lessgtr$ on ob($\mathcal{C}$) by $X \lessgtr Y$ iff $X < Y$ and $Y < X$.

Definition 1.2 A generalized based category $\mathcal{C}$ is a category with coproducts, pullbacks, and final objects (note that this also implies the existence of products and initial object (see [12])) satisfying:

(a) The two squares in the commutative diagram

$$\begin{array}{ccccc} Y_1 & \xrightarrow{j_1} & Z & \xleftarrow{j_2} & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_1 \coprod X_2 & \longleftarrow & X_2 \end{array}$$

are pullbacks iff the upper horizontal arrow represents Z as a coproduct of Y_1 and Y_2.

(b) If $\overline{\mathcal{C}}$ is the full subcategory of $\mathcal{C}$ consisting of indecomposable objects of $\mathcal{C}$ and $X \in$ ob($\overline{\mathcal{C}}$), then $\mathrm{End}_{\overline{\mathcal{C}}}(X) = \mathrm{Aut}_{\overline{\mathcal{C}}}(X)$, i.e., $\overline{\mathcal{C}}$ is an 'EI' category in the terminology of [10] 7.6; [6] or [11].

(c) Any $\mathcal{C}$−object X can be written as a coproduct $X = \coprod_{j\in J} T_j$ where J is an index set and $T_j \in \overline{\mathcal{C}}$.

Definition 1.3 Let $\overline{\mathcal{C}}$ be a full subcategory of $\mathcal{C}$ consisting of indecomposable objects and assume that there is a set $I_{\overline{\mathcal{C}}}$ of representatives of the isomorphism classes of $\overline{\mathcal{C}}$-objects.

Remark 1.4 (i) The definition 1.2 above is more general than the one in [10] 9.4.1 because we do not insist that coproducts be taken over finite sets of objects, that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is finite or even that $I_{\overline{\mathcal{C}}}$ be finite. We reserve the terminology based category for the category defined in [10], 9.4.1.

(ii) Note that for $X, X' \in I_{\overline{\mathcal{C}}}$, $X < X' < X$ implies that $X \approx X'$ (i.e. X is isomorphic to X'). So, $I_{\overline{\mathcal{C}}}$ contains precisely one object out of each equivalence class of indecomposable objects.

Example 1.5 (i) Let G be a finite group, $\mathcal{C} = G\text{-}\mathcal{S}\mathrm{et}^f$ = category of finite G-sets, $\overline{\mathcal{C}} = \mathrm{Or}(G)$ the orbit category of G; i.e., $\mathrm{Or}(G) = \{G/H \mid H \leq G\}$. Morphisms in $\overline{\mathcal{C}}$ are G-maps. Note that a G-set S is indecomposable iff $S \approx G/H$ for some subgroup H of G. If S is a G-set, define $U(S) = \{H \leq G \mid S^H \neq \phi\}$. Note that if S and T are G-sets, then $S < T$ iff $U(S) \subset U(T)$ (see [10], Definition 9.1.3 and Theorem 9.1.1). $\mathcal{C}$ has finite coproducts (disjoint union $\dot{\bigcup}$), products '×', final object $* = G/G$ and initial object ϕ. In this case, any X in ob($\mathcal{C}$) has the form $X = \dot{\bigcup}_{j\in J} (G/H_j)$ where J is finite (i.e., $I_{\overline{\mathcal{C}}}$ is finite) and for any X,Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a finite set. So, $\mathcal{C}$ is a based category.

(ii) Let G be a discrete group and $\mathcal{C}$ = the category of proper discrete G-spaces. Note that X being a proper discrete G-space implies that all the stabilizer groups G_x are finite for all $x \in X$. Let $\mathscr{F}$ be the family of finite subgroups of G and $\overline{\mathcal{C}} = \mathrm{Or}_{\mathscr{F}}(G) = \{G/K \mid K \in \mathscr{F}\}$. $\mathrm{Hom}_{\overline{\mathcal{C}}}(G/K,G/L) = \{G\text{-maps} : G/K - G/L\}$. Note that any $x \in \mathcal{C}$ has the form $X = \dot{\bigcup}_{x\in X} G/G_x$, G_x finite where G_x is the stabilizer of $x \in X$. We also have $\mathrm{Hom}_{\overline{\mathcal{C}}}(G/K,G/K) \approx \mathrm{Aut}(G/K) \approx N_G(K)/K$. $\mathcal{C}$ is a generalized based category with basis $I_{\overline{\mathcal{C}}} = \{G/H \mid$ one H from each conjugacy class of a finite subgroup of $G\}$.

(iii) Let G be a profinite group and $\mathcal{C} = G\text{-}\mathcal{S}\mathrm{et}^f$ the category of finite G-sets. It is a based category where $\overline{\mathcal{C}} = \{G/H \mid H$ open sub group of $G\}$. Note that each G/H in ob($\overline{\mathcal{C}}$) is finite and that $G = \underset{\overleftarrow{H \text{ open}}}{\lim} \lim G/H$. Also any $X \in \mathcal{C}$ has the form $X = \dot{\bigcup} G/H$. If the G-sets are non-finite, then $\mathcal{C}$ is a generalized based category.

2. Generalized Bredon Coefficient Systems and Cohomology

2.1. Generalized Bredon Coefficient Systems

Definition 2.1.1 Let $\mathcal{C}$ be a generalized based category and $\overline{\mathcal{C}}$ the full subcategory of indecomposable objects. Let $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. A generalized Bredon coefficient system is a contravariant functor $\mathcal{A} \to R\text{-}\mathcal{M}\text{od}$ where R is a commutative ring with identity and $R\text{-}\mathcal{M}\text{od}$ is the category of left R-modules. We denote by $\mathcal{B}_R(\mathcal{A})$ the category of Bredon coefficient systems for $\mathcal{A}$.

Our goal in this section 2 is to construct a cohomology theory which translates into Bredon cohomology for each of the examples in 1.5.

2.1.2

In the notation of 2.1.1, $\mathcal{B}_R(\mathcal{A})$ is an Abelian category. Note that for $M, M' \in \text{ob}(\mathcal{B}_R(\mathcal{A}))$, $\ker(M \to M')$ is defined as an object of $\mathcal{B}_R(\mathcal{A})$ such that $\ker(M(X) \to M'(X)) \in \text{ob}(R\text{-}\mathcal{M}\text{od})\ \forall\ X \in \text{ob}(\mathcal{A})$. Define $\text{coker}(M \to M')$ analogously.

A sequence $M' \to M \to M''$ is exact in $\mathcal{B}_R(\mathcal{C})$ iff $M'(X) \to M(X) \to M''(X)$ is exact in $R\text{-}\mathcal{M}\text{od}$ for any $X \in \text{ob}(\mathcal{C})$. Hence, as in any Abelian category, we can define cochain complexes, resolutions, etc. Note also that if $X = \coprod_{j \in J} T_j$ and $M \in \mathcal{B}_R(\mathcal{C})$ then $M(X) = \coprod_{j \in J} \overline{M}(T_j)$ where $\overline{M} = M|_{\overline{\mathcal{C}}}$, and that $\mathcal{B}_R(\mathcal{C})$ is an Abelian category.

2.2. Construction of Typical Projective Objects in $\mathcal{B}_R(\overline{\mathcal{C}})$ and $\mathcal{B}_R(\mathcal{C})$

2.2.1

Let $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. Recall that an object $P \in \mathcal{B}_R(\mathcal{A})$ is said to be projective if $\text{Hom}_{\mathcal{B}_R(\mathcal{C})}(P, -)$; $\mathcal{B}_R(\mathcal{A}) \to R\text{-}\mathcal{M}\text{od}$ is exact.

2.2.2

If $M \in \text{ob}(\mathcal{B}_R(\mathcal{C}))$, put $\overline{M} = M|_{\overline{\mathcal{C}}}$. Let $T \in \text{ob}(\overline{\mathcal{C}})$. Define $\overline{P}_T : \overline{\mathcal{C}} \to R\text{-}\mathcal{M}\text{od}$ by $\overline{P}_T = R\text{Hom}_{\overline{\mathcal{C}}}(T, T') :=$ the R-module generated by $\text{Hom}_{\overline{\mathcal{C}}}(T, T')$.

So, let $f : \overline{P}_T \to \overline{M}$ be a morphism in $\mathcal{B}_R(\overline{\mathcal{C}})$. Then $f(T') : R\text{Hom}_{\overline{\mathcal{C}}}(T', T) \to \overline{M}(T')$. Put $T' = T$. Then we have $R\text{Hom}_{\mathcal{C}}(T, T) \to \overline{M}(T)$. Now by evaluating

$R\mathrm{Hom}_{\overline{\mathcal{C}}}(T,T) \to \overline{M}(T)$ at the identity $1 \in \mathrm{Hom}_{\mathcal{C}}(T,T)$ we get $f(T)1 := \mathrm{ev}_T(f) \in \overline{M}(T)$.

Theorem 2.2.3 *$ev_T : Hom_{\mathcal{B}_R(\overline{\mathcal{C}})}(\overline{P}_T, \overline{M}) \to \overline{M}(T)$ is bijective.*

Proof: Note that by definition 1.2 (b), $\mathrm{Hom}_{\overline{\mathcal{C}}}(T,T) = \mathrm{Aut}_{\overline{\mathcal{C}}}(T)$ is a group and so $\overline{M}(T)$ is a module over the group ring $R\mathrm{Aut}_{\overline{\mathcal{C}}}(T)$. So, every $z \in \overline{M}(T)$ yields a unique $\phi(z) : \overline{P}_T(T) \to \overline{M}(T) : 1 \to z$. Now, $\phi(z)$ extends to a morphism $f : \overline{P}_T \to \overline{M}$ as follows. For any $T' \in \mathrm{ob}(\overline{\mathcal{C}})$, define $f(T') : \overline{P}_T(T') = R\mathrm{Hom}_{\overline{\mathcal{C}}}(T',T) \to \overline{M}(T')$ by $\delta \in \mathrm{Hom}_{\overline{\mathcal{C}}}(T',T)$ goes to $\overline{M}(\delta)\varphi(z)1$ and extend linearly to $R\mathrm{Hom}_{\overline{\mathcal{C}}}(T',T)$. □

2.2.4 $\overline{P}_T$ IS A PROJECTIVE OBJECT IN $\mathcal{B}_R(\overline{\mathcal{C}})$

Since by theorem 2.2.3, $\mathrm{ev}_T : \mathrm{Hom}_{\mathcal{B}_R(\overline{\mathcal{C}})}(\overline{P}_T, \overline{M}) \to \overline{M}(T)$ is bijective, one easily checks that $\mathrm{Hom}_{\mathcal{B}_R(\overline{\mathcal{C}})}(\overline{P}_T, -)$ maps a short exact sequence $\overline{M}' \to \overline{M} \to \overline{M}'' \in \mathcal{B}_R(\overline{\mathcal{C}})$ to a short exact sequence $\overline{M}'(T') \to \overline{M}(T) \to \overline{M}''(T')$ in R-$\mathcal{M}$od. So $\overline{P}_T$ is projective.

2.2.5 EXTENSION TO A PROJECTIVE OBJECT P_X IN $\mathcal{B}_R(\mathcal{C})$

Note that $\overline{\mathcal{C}}$ is a full subcategory of $\mathcal{C}$ and so $\mathrm{Hom}_{\mathcal{B}_R(\overline{\mathcal{C}})}(\overline{P}_T, \overline{M}) = \mathrm{Hom}_{\mathcal{B}_R(\mathcal{C})}(\overline{P}_T, \overline{M})$. Also, any $\overline{M} : \overline{\mathcal{C}} \to R$-$\mathcal{M}$od can be extended to some $M : \mathcal{C} \to R$-$\mathcal{M}$od since if $X = \coprod_{j \in J} T_j$ is in $\mathrm{ob}(\mathcal{C})$ and $T_j \in \mathrm{ob}(\overline{\mathcal{C}})$, we can define $M : \mathcal{C} \to R$-$\mathcal{M}$od by $M(X) = \coprod_{j \in J} \overline{M}(T_j)$.

Note that each $T \in \mathrm{ob}(\overline{\mathcal{C}})$ yields a projective object $\overline{P}_T$ in $\mathcal{B}_R(\overline{\mathcal{C}})$. Now, for any $X = \coprod_{j \in J} T_j$, define $P_X \in \mathrm{ob}(\mathcal{B}_R(\mathcal{C}))$ by: $P_X(-) = R\mathrm{Hom}_{\mathcal{C}}(-,X) = R\mathrm{Hom}_{\mathcal{C}}(-, \coprod_{j \in J} T_j) \cong \coprod_{j \in J} P_{T_j}(-)$ where each $P_{T_j} \in \mathcal{B}_R(\mathcal{C})$ is the extension of $\overline{P}_{T_j}$. Hence, P_X is projective in $\mathcal{B}_R(\mathcal{C})$. So, each object X in $\mathrm{ob}(\mathcal{C})$ defines a projective object P_X in $\mathcal{B}_R(\mathcal{C})$.

2.3. $\mathcal{B}_R(\overline{\mathcal{C}})$ and $\mathcal{B}_R(\mathcal{C})$ have enough projectives

In this subsection, we prove the following theorem.

Theorem 2.3.1 *Let $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. Then $\mathcal{B}_R(\mathcal{A})$ has enough projectives; that is, for any M in $\mathcal{B}_R(\mathcal{A})$, there exists a projective object P in $\mathcal{B}_R(\mathcal{A})$ and an epimorphism $P \to M$.*

Proof: To prove the result for $\mathcal{B}_R(\mathcal{C})$, it suffices to prove it for $\mathcal{B}_R(\overline{\mathcal{C}})$. To see this, suppose that we are given any $X = \coprod_{j\in J} T_j$ in $\text{ob}(\mathcal{C})$ where each $T_j \in \text{ob}(\overline{\mathcal{C}})$, and any $M \in \text{ob}(\mathcal{B}_R(\mathcal{C}))$. Put $\overline{M} = M|_{\overline{\mathcal{C}}}$. Then there exists an epimorphism $\overline{P} \to \overline{M}$ for some projective $\overline{P}$ in $\mathcal{B}_R(\overline{\mathcal{C}})$. Then P defined by $P(X) = \coprod_{j\in J} \overline{P}(T_j)$ is a projective object in $\mathcal{B}_R(\mathcal{C})$ and the morphism $P \to M$ induced by $\overline{P} \to \overline{M}$ is an epimorphism.

Now, for $T \in \mathcal{B}_R(\overline{\mathcal{C}})$, define $\delta_T(T) : \left(\coprod_{z\in\overline{M}(T)} \overline{P}_T\right)(T) \xrightarrow{\varphi(z)} \overline{M}(T)$ by putting the component of $\delta_T(T)$ corresponding to $z \in \overline{M}(T)$ equal to $\phi(z) : \overline{P}_T(T) \to \overline{M}(T) : 1 \to z$ as in the proof of 2.2.3. Then $\delta_T(T)$ is onto, and extends to a unique morphism $\delta_T : \coprod \overline{P}_T \to \overline{M}$. We thus have a morphism $\delta : \coprod_{T\in\overline{\mathcal{C}}} (\coprod_{M(T')} \overline{P}_{T'}) \to \overline{M}$. Then $\delta : \coprod_{T\in\overline{\mathcal{C}}} (\coprod_{M(T')} \overline{P}_{T'}) \to \overline{M}$ is an epimorphism where $\overline{P}$ is projective in $\mathcal{B}_R(\overline{\mathcal{C}})$.

□

Remark 2.3.2 It follows from theorem 2.3.1 that if we put $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$, then given $M \in \text{ob}(\mathcal{B}_R(\mathcal{A}))$ we have a projective resolution $P_*(M) \to M$. If N is any other object of $\mathcal{B}_R(\mathcal{A})$, we have cohomology groups

$$Ext^i(M,N) = H^i(\text{Hom}_{\mathcal{B}_R(\mathcal{A})}(P_*(M),N)).$$

2.4. Some Applications

2.4.1

(1) Let G be a discrete group and $\mathcal{C}$ the category of proper discrete G-spaces. If $\mathcal{F}$ is the collection of finite subgroups of G, put $\overline{\mathcal{C}} = \text{Or}_{\mathcal{F}}(G) = \{G/H \mid H \in \mathcal{F}\}$.

If $R = \mathbb{Z}$, and $\overline{M} = \overline{\mathbb{Z}}$, the constant functor: $\text{Or}_{\mathcal{F}}(G) \to R\text{-}\mathcal{M}\text{od}$, one could then recover classical Bredon cohomology of G where $H^i(G,\overline{N}) = \text{Ext}^i(\overline{\mathbb{Z}},\overline{N})$ and $\overline{N} \in \text{ob}(\mathcal{B}_{\mathbb{Z}}(\text{Or}_{\mathcal{F}} G))$.

Note that since any $X \in \text{ob}(G\text{-}\mathcal{S}\text{et}^d)$ can be written as $X = \dot{\bigcup}_{H\in\mathcal{F}} G/H$, the cohomology theory in $\text{ob}(\mathcal{B}_R(\text{Or}_{\mathcal{F}}(G)))$ can be extended to $\text{Ext}^i(M,N) = H^i(\text{Hom}_{\mathcal{B}_R(\mathcal{C})}(P_*(M),N))$ where $\overline{M} = M|_{\mathcal{B}_R(\overline{\mathcal{C}})}$, (see 2.2.5 and theorem 2.3.1), and $N \in \text{ob}(\mathcal{B}_R(G\text{-}\mathcal{S}\text{et}^d))$.

(2) Let G be a finite group, $\mathcal{C} = G\text{-}\mathcal{S}\text{et}^f :=$ the category of finite G-sets and $\mathcal{C} = \text{Or}(G) = \{G/H \mid H \leq G\}$ the orbit category of G. Any $X \in \mathcal{C}$ has the form $X = \dot{\bigcup}_{H\leq G} G/H$ (finite disjoint union) and $\text{End}_{\overline{\mathcal{C}}}(G/H) = \text{Aut}(G/H)$ is a finite group. By following the procedure in 2.2, 2.3, we obtain a projective object $\overline{P}_T$ where $T = G/H$. So, for any $\overline{M},\overline{N} \in \mathcal{B}_R$ (or G), this procedure would yield

Bredon cohomology groups $\mathrm{Ext}^i(\overline{M},\overline{N})$, $\mathrm{Ext}^i(M,N)$, as well as $H^i(G,\overline{N})$ of G where $\overline{M} = M|_{\mathrm{Or}(G)}$, $\overline{N} = N|_{\mathrm{Or}(G)}$.

(3) Let G be a profinite group, $\mathcal{C}$ the category of finite G-sets and $\overline{\mathcal{C}} = \{G/H \mid H$ open subgroup of $G\}$. In this case also, one can construct the cohomology groups $\mathrm{Ext}^i(\overline{M},\overline{N})$ and $\mathrm{Ext}^i(M,N)$ as well as Bredon cohomology groups $H^i(G,\overline{N})$ for G, by following the procedure in 2.2 and 2.3.

3. Higher K-theory of $\mathcal{B}_R(\mathcal{C})$

3.1. $K_n(\mathcal{P}(\mathcal{B}_R(\mathcal{C}))$ $K_n(\mathcal{M}(\mathcal{B}_R(\mathcal{C}))$

3.1.1

Let $\mathcal{C}$ be a generalized based category and $\overline{\mathcal{C}}$ the full subcategory of indecomposable objects of $\mathcal{C}$. Recall from [10] 7.6, [11], [6] that an EI category is a small category in which every endomorphism is an isomorphism. So, from axiom (b) of definition 1.2, we have that $\overline{\mathcal{C}}$ is an EI category. So the results in [10] 7.6, [11] or [6] hold for $\mathcal{B}_R(\overline{\mathcal{C}})$. In this section, we extend some of these results to $\mathcal{B}_R(\mathcal{C})$ while obtaining some new ones.

Definition 3.1.2 Let $\mathcal{C}$, $\overline{\mathcal{C}}$ be as in 3.1.1 and $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. An ob($\mathcal{A}$)-set is a pair (L,β) where L is a set and $\beta : L \to \mathrm{ob}(\mathcal{A})$ a set map. Thus $L = \{\beta^{-1}(a) \mid a \in \mathrm{ob}(\mathcal{A})\}$.

Note that

(i) an object M in $\mathcal{B}_R(\mathcal{A})$ has an underlying ob($\mathcal{A}$)-set also denoted by M. This ob($\mathcal{A}$)-set is a pair (L,β) where L is the disjoint union of the sets $M(X)$ as X ranges over ob($\mathcal{A}$) and β assigns to each element x of L the object X such that $M(X)$ contains x.

(ii) We could also interpret ob($\mathcal{A}$)-sets as a functor : ob($\mathcal{A}$) $\to \mathcal{S}$ets and a map between ob($\mathcal{A}$)-sets as a natural transformation.

Definition 3.1.3 Let $\mathcal{C}$, $\overline{\mathcal{C}}$ be as in 3.1.1, and put $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. Let B be a set and $\beta : B \to \mathrm{ob}(\mathcal{C})$ be a map. Then an object $M \in \mathcal{B}_R(\mathcal{A})$ with $b \in M\beta(b)$ $\forall$ $b \in B$ is said to be free with base B if for any $N \in \mathcal{B}_R(\mathcal{C})$ and any map $f : B \to N$ with $f(b) \in N(\beta(b))$ for all $b \in B$ there exists exactly one $\mathcal{B}_R(\mathcal{C})$-map $F : M \to N$ extending f.

Definition 3.1.4 Let $\mathcal{C}$, $\overline{\mathcal{C}}$ be as in 3.1.1, and put $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. An ob($\mathcal{A}$)-set (N,β) is said to be finite if N is finite. If S is an (N,β) subset of an object $M \in \mathrm{ob}(\mathcal{B}_R(\mathcal{A}))$, define span S as the smallest $\mathcal{B}_R(\mathcal{A})$-subobject of M containing S. Say that M is finitely generated if N is finite and span $S = M$.

Definition 3.1.5 Let $\mathcal{C}$, $\overline{\mathcal{C}}$ be as in 3.1.1, and put $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. Let $\mathcal{P}(\mathcal{B}_R(\mathcal{A}))$ be the category of finitely generated projective objects in $\mathcal{B}_R(\mathcal{A})$. Then $\mathcal{P}(\mathcal{B}_R(\mathcal{A}))$ is an exact category in the sense of Quillen [14] and we denote $K_n(\mathcal{P}(\mathcal{B}_R(\mathcal{A})))$ by $K_n(R\mathcal{A})$ for all $n \geq 0$.

Definition 3.1.6 Let $\mathcal{C}$, $\overline{\mathcal{C}}$ be as in 3.1.1, and put $\mathcal{A} = \mathcal{C}$ or $\overline{\mathcal{C}}$. Let R be a commutative Noetherian ring, and $\mathcal{M}(\mathcal{B}_R(\mathcal{A}))$ the category of finitely generated objects in $\mathcal{B}_R(\mathcal{A})$. Then $\mathcal{M}(\mathcal{B}_R(\mathcal{A}))$ is an exact category in the sense of Quillen [14] and we denote $K_n(\mathcal{M}(\mathcal{B}_R(\mathcal{A})))$ by $G_n(R\mathcal{A})$.

We now record the following results due to W. Luck. (see [11]).

Theorem 3.1.7 *(a) Let R be a commutative ring with identity and $\mathcal{D}$ an EI-category. Then $K_n(R\mathcal{D}) \approx \bigoplus_{T \in I_{\mathcal{D}}} K_n(R(Aut(T))) \ \forall\, n \geq 1$.*

(b) Let R be a commutative Noetherian ring and $\mathcal{D}$ an EI-category. Then $G_n(R\mathcal{D}) \approx \bigoplus_{T \in I_{\mathcal{D}}} G_n(R(Aut(T))) \ \forall\, n \geq 1$.

Note that in (a), (b), $R(Aut(T))$ is the group ring RG where $G = Aut(T)$ and $I_{\mathcal{D}}$ is the set of representatives of isomorphism classes of $\mathcal{D}$-objects.

Proof: See [10], [6] for a sketch of the proofs and [11] for full proofs of (a) and (b). □

We now have the following results.

Theorem 3.1.8 *Let $\mathcal{C}$ be a based category and $\overline{\mathcal{C}}$ the full subcategory of indecomposable objects in $\mathcal{C}$.*

(a) If R is any commutative ring with identity, then

$$K_n(R\mathcal{C}) \approx \bigoplus \left(\bigoplus_{T \in I_{\overline{\mathcal{C}}}} K_n(R(Aut(T))) \right) \ \forall\, n \geq 0.$$

(b) If R is a commutative Noetherian ring then

$$G_n(R\mathcal{C}) \approx \bigoplus \left(\bigoplus_{T \in I_{\overline{\mathcal{C}}}} G_n(R(Aut(T))) \right) \ \forall\, n \geq 0.$$

Note: In (a), (b) above $R(Aut(T))$ is the group ring RG where $G = Aut(T)$).

Proof: Since any $X \in \mathrm{ob}(\mathcal{C})$ has the form $X = \coprod_{j \in J} T_j$ (J some finite index set), we have $\mathcal{B}_R(\mathcal{C}) = \coprod_{j \in J} \mathcal{B}_R(\overline{\mathcal{C}})$. So

$$\mathcal{P}(\mathcal{B}_R(\mathcal{C})) \approx \coprod_{j \in J} \mathcal{P}(\mathcal{B}_R(\overline{\mathcal{C}}))$$

where each j^{th} component$= \mathcal{P}(\mathcal{B}_R(\overline{\mathcal{C}}))$ and

$$\mathcal{M}(\mathcal{B}_R(\mathcal{C}))\coprod_{j\in J}\mathcal{M}(\mathcal{B}_R(\overline{\mathcal{C}})).$$

Hence by Quillen's results (see [14]),

$$K_n(\mathcal{P}(\mathcal{B}_R(\mathcal{C}))) := K_n(R\mathcal{C}) \approx \bigoplus_{j\in J} K_n(\mathcal{P}(\mathcal{B}_R(\overline{\mathcal{C}}))) = \bigoplus_{j\in J}(K_n(R\overline{\mathcal{C}})).$$

Similarly,

$$K_n(\mathcal{M}(\mathcal{B}_R(\mathcal{C}))) := G_n(R\mathcal{C}) \approx \bigoplus G_n(R\overline{\mathcal{C}}).$$

However, by theorem 3.1.7

$$K_n(R\overline{\mathcal{C}}) \approx \bigoplus_{T\in I_{\overline{\mathcal{C}}}} K_n(R(\mathrm{Aut}(T)))$$

and

$$G_n(R\overline{\mathcal{C}}) \approx \bigoplus_{T\in I_{\overline{\mathcal{C}}}} G_n(R(\mathrm{Aut}(T))) \text{ since } \overline{\mathcal{C}} \text{ is an } EI\text{-category.}$$

Hence

$$K_n(R\mathcal{C}) \approx \bigoplus\left(\bigoplus_{T\in I_{\overline{\mathcal{C}}}} K_n(R(\mathrm{Aut}(T)))\right)$$

and

$$G_n(R\mathcal{C}) \approx \bigoplus\left(\bigoplus_{T\in I_{\overline{\mathcal{C}}}} G_n(R(\mathrm{Aut}(T)))\right).$$

□

We now prove the following application of theorem 3.1.8.

Theorem 3.1.9 *Let $\mathcal{C}$ be a based category and R the ring of integers in a number field F. Then for all $n \geq 1$:*

1. *(a) $K_{2n-1}(R\mathcal{C})$ are finitely generated Abelian groups.*

   (b) $K_{2n}(R\mathcal{C})$ are finite groups.

2. *(a) $G_{2n-1}(R\mathcal{C})$ are finitely generated Abelian groups.*

   (b) $G_{2n}(R\mathcal{C})$ are finite groups.

3. *$\mathrm{rank}\, K_{2n-1}(R\mathcal{C}) = \mathrm{rank}\, G_{2n-1}(R\mathcal{C}) = \mathrm{rank}\, K_{2n-1}(F\mathcal{C})\ \forall\ n \geq 2.$*

Proof: 1 and 2. Since $\mathcal{C}$ is a based category, we have that $I_{\overline{\mathcal{C}}}$ is a finite set (see [10] 9.4.1 (ii)) and for each $T \in I_{\overline{\mathcal{C}}}$, $\text{Aut}(T)$ is a finite group. From 3.1.8, we have $K_n(R\mathcal{C}) \approx \bigoplus\left(\bigoplus_{T\in I_{\overline{\mathcal{C}}}} K_n(R(\text{Aut}(T)))\right)$ and $G_n(R\mathcal{C}) \approx \bigoplus\left(\bigoplus_{T\in I_{\overline{\mathcal{C}}}} G_n(R(\text{Aut}(T)))\right)$ where $R(\text{Aut}(T))$ is the group ring RG and $G = \text{Aut}(T)$ is finite. So, results 1(b) and 2(b) follow from [10] theorem 7.2.7, since for any finite group G, $K_{2n}(RG)$ and $G_{2n}(RG)$ are finite groups for all $n \geq 1$. Moreover, 1(a) and 2(a) follow from [10] theorem 7.1.11 and 7.1.13 where it is proved that $K_{2n-1}(RG)$, $G_{2n-1}(RG)$ are finitely generated Abelian groups for all $n \geq 1$.

3. As already observed above, $I_{\overline{\mathcal{C}}}$ is a finite set and for any $T \in \text{ob}(\overline{\mathcal{C}})$, $\text{Aut}(T)$ is a finite group.

$$K_{2n-1}(R\mathcal{C}) = \bigoplus\left(\bigoplus_{T\in I_{\mathcal{C}}} K_{2n-1}(R(\text{Aut}(T)))\right)$$

$$G_{2n-1}(R\mathcal{C}) = \bigoplus\left(\bigoplus_{T\in I_{\mathcal{C}}} G_{2n-1}(R(\text{Aut}(T)))\right)$$

$$K_{2n-1}(F\mathcal{C}) = \bigoplus\left(\bigoplus_{T\in I_{\mathcal{C}}} K_{2n-1}(F(\text{Aut}(T)))\right).$$

Now by [10], theorem 7.2.1 or [7] $\text{rank}\, K_{2n-1}(R(\text{Aut}(T))) = \text{rank}\, G_{2n-1}(R(\text{Aut}(T)))$ $= \text{rank}\, K_{2n-1}(F(\text{Aut}(T)))\ \forall\ n \geq 2$. Hence $\text{rank}\, K_{2n-1}(R\mathcal{C}) = \text{rank}\, G_{2n-1}(R\mathcal{C}) =$ $\text{rank}\, K_{2n-1}(F\mathcal{C})$ as required. □

Corollary 3.1.10 *Let G be a finite or profinite group and $\mathcal{C} = G\text{-}\mathcal{S}et^f$ Let R be the ring of integers in a number field F. Then for all $n \geq 1$:*

1. *(a) $K_{2n-1}(R(G\text{-}\mathcal{S}et^f))$ are finitely generated Abelian groups.*

   (b) $K_{2n}(R(G\text{-}\mathcal{S}et^f))$ are finite Abelian groups.

2. *(a) $G_{2n-1}(R(G\text{-}\mathcal{S}et^f))$ are finitely generated Abelian groups.*

   (b) $G_{2n}(R(G\text{-}\mathcal{S}et^f))$ are finite Abelian groups.

3. *$\text{rank}\, G_{2n-1}(R(G\text{-}\mathcal{S}et^f)) = \text{rank}\, K_{2n-1}(R(G\text{-}\mathcal{S}et^f)) = \text{rank}\, K_{2n-1}(F(G\text{-}\mathcal{S}et^f))$.*

The corollary follows from putting $\mathcal{C} = G\text{-}\mathcal{S}\text{et}^f$ in theorem 3.1.8.

3.2. $SK_n(R\mathcal{C})$; $SK_n(\widehat{R}_{\underline{p}}\mathcal{C})$; $SG_n(R\mathcal{C})$; $SG_n(\widehat{R}_{\underline{p}}\mathcal{C})$

Definition 3.2.1 Let $\mathcal{C}$ be a based category and R a Dedekind domain with quotient field F. It is easily checked that the inclusion map $R \to F$ induces exact functors $\mathcal{P}(\mathcal{B}_R(\mathcal{C})) \to \mathcal{P}(\mathcal{B}_F(\mathcal{C}))$ (resp. $\mathcal{M}(\mathcal{B}_R(\mathcal{C})) \to \mathcal{M}(\mathcal{B}_F(\mathcal{C}))$) and hence Abelian group homomorphisms $K_n(\mathcal{P}(\mathcal{B}_R(\mathcal{C}))) \xrightarrow{\phi} K_n(\mathcal{P}(\mathcal{B}_F(\mathcal{C})))$ (resp. $K_n(\mathcal{M}(\mathcal{B}_R(\mathcal{C}))) \to K_n(\mathcal{M}(\mathcal{B}_F(\mathcal{C}))))$ that is $K_n(R\mathcal{C}) \xrightarrow{\phi} K_n(F\mathcal{C})$ (resp. $G_n(R\mathcal{C}) \xrightarrow{\delta} G_n(F\mathcal{C}))$ for all $n \geq 0$.

Define

$$SK_n(R\mathcal{C}) = \ker\phi$$
$$SG_n(R\mathcal{C}) = \ker\delta.$$

Note:
We shall be interested in the cases when R is the ring of integers in a number field F and $\widehat{R}_{\underline{p}}$ is the completion of R at a prime ideal $\underline{p}$.

We now prove the following result.

Theorem 3.2.2 *Let $\mathcal{C}$ be a based category, R the ring of integers in a number field F, and $\widehat{R}_{\underline{p}}$ (resp. $\widehat{F}_{\underline{p}}$) the completion of R (resp. F) at a prime ideal $\underline{p}$ of R. Then for all $n \geq 1$:*

1. *(a) $SG_n(R\mathcal{C}) = 0 = SG_n(\widehat{R}_{\underline{p}}\mathcal{C})$.*

   (b) Hence $G_{2n-1}(R\mathcal{C}) \approx G_{2n-1}(F\mathcal{C})$ and $G_{2n-1}(\widehat{R}_{\underline{p}}\mathcal{C}) \approx G_{2n-1}(\widehat{F}_{\underline{p}}\mathcal{C})$. Moreover, $G_{2n-1}(F\mathcal{C})$ is a finitely generated Abelian group.

2. *$SK_n(R\mathcal{C})$ and $SK_n(\widehat{R}_{\underline{p}}\mathcal{C})$ are finite groups.*

3. *Cokernel of ϕ is torsion.*

Proof: 1. (a) First note that in theorem 3.1.8, $I_{\overline{\mathcal{C}}}$ is a finite set since $\mathcal{C}$ is a based category. Thus

$$K_n(R\mathcal{C}) = \bigoplus\left(\bigoplus_{T\in\overline{\mathcal{C}}} K_n(R(\mathrm{Aut}(T)))\right) \tag{I}$$

$$G_n(R\mathcal{C}) = \bigoplus\left(\bigoplus_{T\in\overline{\mathcal{C}}} G_n(R(\mathrm{Aut}(T)))\right) \tag{II}$$

where all direct sums are over finite index sets. Also,

$$SK_n(R\mathcal{C}) \approx \bigoplus\left(\bigoplus_{T\in\overline{\mathcal{C}}} SK_n(R(\mathrm{Aut}(T)))\right) \tag{III}$$

and

$$SG_n(R\mathcal{C}) \approx \bigoplus\left(\bigoplus_{T\in\overline{\mathcal{C}}} SG_n(R(\mathrm{Aut}(T)))\right) \tag{IV}$$

where $R(\mathrm{Aut}(T))$ is the group ring over the finite group $\mathrm{Aut}(T)$. But by [10] Corollary 7.1.3, $SG_n(RG) = 0$ for $n \geq 1$ and any finite group G. Hence $SG_n(R\mathcal{C}) = 0 \ \forall\ n \geq 1$.

That $SG_n(\widehat{R}_{\underline{p}}\mathcal{C}) = 0$ follows from the fact that (IV) above holds when R is replaced by $\widehat{R}_{\underline{p}}$ and from the fact that $SG_n(\widehat{R}_{\underline{p}}(\mathrm{Aut}(T))) = 0$ since $SG_n(\widehat{R}_{\underline{p}}G) = 0$ for any finite group G (see [8] theorem 1.9(i)).

(b) Since by [10] Corollary 7.1.3, $SG_n(RG) = 0$ for any finite group G, we have by [10] Remarks 8.2.6 (ii) or [8] that

$$G_{2n-1}(RG) \approx G_{2n-1}(FG)\ \forall\ n \geq 2. \tag{V}$$

Now

$$G_{2n-1}(R\mathcal{C}) \approx \bigoplus\left(\bigoplus_{T\in\overline{\mathcal{C}}} G_{2n-1}(R(\mathrm{Aut}(T)))\right). \tag{VI}$$

Also,

$$G_{2n-1}(F\mathcal{C}) \approx \bigoplus\left(\bigoplus_{T\in\overline{\mathcal{C}}} G_{2n-1}(F(\mathrm{Aut}(T)))\right). \tag{VII}$$

Since

$$G_{2n-1}(R(\mathrm{Aut}(T))) \approx G_{2n-1}(F(\mathrm{Aut}(T))) \tag{VIII}$$

by (I), we have

$$G_{2n-1}(R\mathcal{C}) \approx G_{2n-1}(F\mathcal{C}).$$

That $G_{2n-1}(F\mathcal{C})$ is finitely generated follows from the fact that $G_{2n-1}(R\mathcal{C})$ is finitely generated.

2. First observe that $SK_n(R(\mathrm{Aut}(T)))$ is finite for all $n \geq 1$ since $\mathrm{Aut}(T)$ is a finite group (see [10] theorem 7.1.11(ii)). Also, $SK_n(\widehat{R}_{\underline{p}}(\mathrm{Aut}(T))$ is finite by [10] theorem 7.1.11 (iii). So, by (III) above, $SK_n(R\mathcal{C})$ is finite for all $n \geq 1$. Also, (III) holds with R replaced by $\widehat{R}_{\underline{p}}$. Hence $SK_n(\widehat{R}_{\underline{p}}\mathcal{C})$ is finite.

3. Put $G = \mathrm{Aut}(T)$. Then $R(\mathrm{Aut}(T))$ is a group ring over the finite group G and $\mathrm{Coker}(K_n(R(\mathrm{Aut}(T)))) \to K_n(F(\mathrm{Aut}(T)))$ is torsion by [10] theorem 7.2.6 or [6] for all $n \geq 2$. Hence Coker φ is torsion. □

Remark 3.2.3 Note that 3.2.2 applies to $\mathcal{C} = G\text{-}\mathcal{S}\mathrm{et}^f$ where G is a finite or profinite group.

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