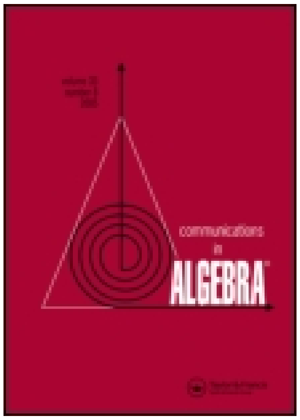

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Higher Algebraic *K*-Theory of *p*-Adic Orders and Twisted Polynomial and Laurent Series Rings Over *p*-Adic Orders

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HIGHER ALGEBRAIC K-THEORY OF p-ADIC ORDERS AND TWISTED POLYNOMIAL AND LAURENT SERIES RINGS OVER p-ADIC ORDERS

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Let F be a p-adic field (i.e., any finite extension of $\widehat{\mathbb{Q}}_p$), R the ring of integers of F, Λ any R-order in a semisimple F-algebra Σ, $\alpha : \Lambda \to \Lambda$ an R-automorphism of Λ, $T = \langle t \rangle$, the infinite cyclic group, $\Lambda_\alpha[t]$, the α-twisted polynomial ring over Λ and $\Lambda_\alpha[T]$, the α-twisted Laurent series ring over Λ. In this article, we study higher K-theory of Λ, $\Lambda_\alpha[t]$, and $\Lambda_\alpha[T]$.

More precisely, we prove in Section 1 that for all $n \geq 1$, $SK_{2n-1}(\Lambda)$ is a finite p-group if Σ is a direct product of matrix algebra over fields, in partial answer to an open question whether $SK_{2n-1}(\widehat{\mathbb{Z}}_p G)$ is a finite p-group if G is any finite group. So, the answer is affirmative if $\widehat{\mathbb{Q}}_p G$ splits.

We also prove that $NK_n(\Lambda; \alpha) := \ker(K_n(\Lambda_\alpha[t]) \to K_n(\Lambda))$ is a p-torsion group and also that for $n \geq 2$ there exists isomorphisms

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T]).$$

Finally, we prove that $NK_n(\Lambda_\alpha[T])$ is p-torsion. Note that if G is a finite group and $\Lambda = RG$ such that $\alpha(G) = G$, then $\Lambda_\alpha[T]$ is the group ring RV where V is a virtually infinite cyclic group of the form $V = G \rtimes_\alpha T$, where α is an automorphism of G and the action of the infinite cyclic group $T = \langle t \rangle$ on G is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$.

Key Words: Farrell–Jones conjecture; Higher algebraic K-theory; Twisted polynomials and Laurent series rings; p-Adic orders in algebras; Virtually infinite cyclic groups.

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INTRODUCTION

Let F be a p-adic field (i.e., any finite extension of $\widehat{\mathbb{Q}}_p$), p an odd prime, R the ring of integers in F (i.e., the integral closure of $\widehat{\mathbb{Z}}_p$ in F), Λ any R-order in a semisimple F-algebra Σ, $\alpha : \Lambda \to \Lambda$ an R-automorphism of Λ, $\Lambda_\alpha[t]$ the α-twisted

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polynomial ring over Λ and $\Lambda_\alpha[T]$ the α-twisted Laurent series ring over Λ (see notes on Notation). It is the aim of this article to study higher K-Theory of Λ, $\Lambda_\alpha[t]$, and $\Lambda_\alpha[T]$. Note that if α is an automorphism of G and the action of the infinite cyclic group $T = \langle t \rangle$ on G is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$, then $\Lambda_\alpha[T] \simeq (RG)_\alpha[T]$ is isomorphic to the group ring RV, where $V = G \rtimes_\alpha T$ is the semidirect product.

Note that this article is the first to study higher K-theory of $\Lambda_\alpha[T]$ which translates into RV in the local situation of R being the ring of integers in a p-adic field. Before now, higher K-theory of $\Lambda_\alpha[T]$ and hence RV has been studied only in global context of R being the ring of integers in a number field, (see [12, 7.5], [13, 14]).

Note that the study of K-theory of $\Lambda_\alpha[T]$ and RV in the global case of R being the ring of integers in a number field is intimately connected with the study of Farrell–Jones conjecture [3, 4, 12], which asserts roughly that K-theory of an arbitrary discrete group H should have as building blocks the K-theory of virtually cyclic subgroups of H. Recall that a group H is virtually cyclic if it is either finite or virtually infinite cyclic; i.e., contains a finite index subgroup of H that is infinite cyclic. Moreover, one type of virtually infinite cyclic group $V = G \rtimes_\alpha T$ is in focus in this article.

The study of higher K-theory in the global context of R being the ring of integers in a number field is facilitated by several finiteness results which exist in this case. For all $n \geq 1$, $K_n(\Lambda), G_n(\Lambda)$ (hence $K_n(RG)$), $G_n(RG)$ are finitely generated, $K_{2n}(\Lambda)$, $G_{2n}(\Lambda)$ (and hence $K_{2n}(RG)$, $G_{2n}(RG)$) are finite groups (see [6, 8, 11, 12]), $G_n(\Lambda_\alpha)[T]$ (and hence $G_n(RV)$) are finitely generated. See [12, 13].

However, in the local situation of R being the ring of integers in a p-adic field, such finite generation results do not hold for $K_n(\Lambda)$, $G_n(\Lambda)$, $G_n(\Lambda_\alpha[T])$, and so we have to get around these problems to nevertheless obtain interesting results.

We now briefly review the results in this article.

Section 1 provides a partial answer to the following open question. See [12, Problem 1].

Given any finite group G, is $SK_{2n-1}(\widehat{\mathbb{Z}}_pG)$ a finite p-group for all $n \geq 1$? An affirmative answer for $n = 1$ was due to Wall (see [17]).

A positive answer for G a finite p-group all $n \geq 1$ is due to Kuku (see [12, Theorem 7.1.17] or [11]). We prove in this article a general result which says that if F is a p-adic field with ring of integer R, and Λ any R-order in a semisimple F-algebra, then for all $n \geq 1$, $SK_{2n-1}(\Lambda)$ is a finite p-group if Σ splits as a product of matrix algebras over fields. So $SK_{2n-1}(RG)$ is a finite p-group if FG splits. We note that in the case proved earlier by this author, namely, $SK_{2n-1}(\widehat{\mathbb{Z}}_pG)$ is a finite p-group, where G is a finite p-group, and p is an ode prime, then $\widehat{\mathbb{Q}}_pG$ indeed splits.

In Section 2 we study $K_n(\Lambda_\alpha[t]), n \geq 1$ and prove that $NK_n(\Lambda, \alpha) := \ker(K_n(\Lambda_\alpha[t]) \to K_n(\Lambda))$ is a p-torsion group. So, for any finite group G, $NK_n(RG; \alpha)$ is a p-torsion group if R is the ring of integers in a p-adic field.

In Section 3, we prove that there exists isomorphisms

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$$

(see Theorem 3).

This is proved in three parts; namely:

(a) $\mathbb{Q} \otimes G_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T])$ (Theorem 4);
(b) $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])$ (Theorem 7); and
(c) $\mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Sigma_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$ (Theorem 8),

where Γ is an α-invariant R-order containing Λ. Note that Γ is right-regular and hereditary. (The proof of the existence of Γ is an adaptation of the proof of Theorem 7.5.2 in [12], for the global situation).

Notes on Notation

For an exact category $\mathscr{C}$, we write $K_n(\mathscr{C})$ for the Quillen higher K-groups $\pi_{n+1}(BQ\mathscr{C})$ for $n \geq 0$ (see [10, 15]).

If A is any ring with identity, we write, for $n \geq 0$, $K_n(A) = K_n(\mathscr{P}(A))$, where $\mathscr{P}(A)$ is the category of finitely generated projective modules over A, and when A is Noetherian, we write $G_n(A)$ for $K_n(\mathscr{M}(A))$, where $\mathscr{M}(A)$ is the category of finitely generated A-modules. If α is an automorphism of ring A, $A_\alpha[T]$ is defined additively by $A_\alpha[T] = A_\alpha[t, t^{-1}] = A[T]$ with multiplication given by $(at^i) \bullet (bt^j) = a\alpha^i(b)t^{i+j}$ for $a, b \in A$. Call $A_\alpha[T]$ the twisted Laurent series ring over A. $A_\alpha[t]$ (resp., $A_\alpha[t^{-1}]$ is the subring of $A_\alpha[T]$ generated by A and t (resp., A and t^{-1}). Call $A_\alpha[t]$ the α-twisted polynomial ring over A. We also have inclusion maps $i : A \to A_\alpha[T]$, $i^+ : A \to A_\alpha[t]$, and $i^- : A \to A_\alpha[t^{-1}]$. The augmentation map $\epsilon : A_\alpha[t] \to A$ induces a group homomorphism $\epsilon_* : K_n(A_\alpha[t]) \to K_n(A)$, and we put $NK_n(A; \alpha) := \ker \epsilon_*$. Since ϵ is split by i^+, we have $K_n(A_\alpha[t]) \simeq K_n(A) \oplus NK_n(A; \alpha)$.

If R is a Dedekind domain with quotient field F, Λ an R-order in a semisimple F-algebra Σ, then for all $n \geq 0$, $SK_n(\Lambda)$ is defined as the kernel of the canonical map $K_n(\Lambda) \longrightarrow K_n(\Sigma)$.

We assume throughout this article that p denotes an odd rational prime.

1. ON A QUESTION ON $SK_{2N-1}(\widehat{\mathbb{Z}}_P G)$, G A FINITE GROUP

1.1.

In this section we provide a partial answer to the following open question (see [12, Problem 1]).

Let G be any finite group, p a rational prime, $\widehat{\mathbb{Q}}_p$ the completion of $\mathbb{Q}$ at p, $\widehat{\mathbb{Z}}_p$ the ring of integers of $\widehat{\mathbb{Q}}_p$. Is $SK_{2n-1}(\widehat{\mathbb{Z}}_p G)$ a finite p-group for all $n \geq 1$?

For $n = 1$, a positive answer is due to Wall (see [16]).

For G a finite p-group, and for all $n \geq 1$, an affirmative answer is due to Kuku (see [12, Theorem 7.1.17] or [11]).

We prove in this section a general result which partially answers this question.

Remark 1. It is a result due to Kuku that for all $n \geq 1$, and for any finite group G, $SK_n(\widehat{\mathbb{Z}}_p G)$ is finite. In fact, he proved a more general result that if F is any p-adic field (i.e., F is any finite extension of $\widehat{\mathbb{Q}}_p$) and R the ring of integers of F (i.e., R is the integral closure of $\widehat{\mathbb{Z}}_p$ in F) Λ, any R-order in a semisimple F-algebra Σ, then $SK_n(\Lambda)$ is finite for all $n \geq 1$ (see [12, Theorem 7.1.11 (iii)] or [8]).

Hence for any finite group G, $SK_n(RG)$ is finite for all $n \geq 1$.

We now prove the following result connected to the Question in 1.1.

Theorem 1. *Let F be a p-adic field with ring of integers R, Λ any R-order in a semisimple F-algebra Σ. Then $SK_{2n-1}(\Lambda)$ is a finite p-group if Σ is a product of matrix algebras over fields, where n is an integer, $n \geq 1$.*

Proof. First note that $\Sigma = \Lambda(\frac{1}{p})$. So, if Γ is a maximal R-order containing Λ, $\Lambda \subset \Gamma \subset \Lambda(\frac{1}{p})$, and so there exists a positive integer s such $p^s\Gamma \subset \Lambda$. So if we put $\underline{q} = p^s\Gamma$, then $\underline{q}$ is an ideal of both Γ and Λ, and so we have a Cartesian square

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Lambda/\underline{q} & \longrightarrow & \Gamma/\underline{q}. \end{array}$$

Since p^s annihilates $\Gamma/\underline{q}$, $\underline{q} \otimes \mathbb{Z}(\frac{1}{p}) = \underline{q} \otimes \mathbb{Z}(\frac{1}{p^s}) \simeq \Gamma \otimes \mathbb{Z}(\frac{1}{p^s})$ is a ring with unit and so, as in [1, 10], there exists a Mayer–Vietoris sequence

$$\begin{aligned} &\to K_{n+1}(\Gamma/\underline{q})\left(\frac{1}{p}\right) \to K_n(\Lambda)\left(\frac{1}{p}\right) \to K_n(\Gamma)\left(\frac{1}{p}\right) \oplus K_n(\Lambda/\underline{q})\left(\frac{1}{p}\right) \\ &\to K_n(\Gamma/\underline{q})\left(\frac{1}{p}\right) \to . \end{aligned} \tag{I}$$

Now $\left(\Gamma/\underline{q}\right)$, $\left(\Lambda/\underline{q}\right)$ are finite rings and so $K_n(\Gamma/\underline{q})$, and $K_n(\Lambda/\underline{q})$ are finite groups (see [12, Theorem 7.1.12] or [7]). If we put $B = \Gamma/\underline{q}$ or $\Lambda/\underline{q}$, then B is a $\mathbb{Z}/p^s$-algebra and so the group $K_n(B; \operatorname{rad} B)$ in the relative sequence

$$K_{n+1}(B/\operatorname{rad} B) \to K_n(B, \operatorname{rad} B) \to K_n(B) \to K_n(B/\operatorname{rad} B) \to \cdot \tag{II}$$

is a finite p-group (see [18, Corollary 5.4]).

By tensoring the sequence (II) with $\mathbb{Z}(\frac{1}{p})$, we have that $K_n(B)(\frac{1}{p}) \simeq K_n(B/\operatorname{rad} B)(\frac{1}{p})$. Now since $B/\operatorname{rad} B$ is a finite semisimple (regular) ring, then for $r \geq 1, K_{2r}(B/\operatorname{rad} B) \simeq G_{2r}(B/\operatorname{rad} B) = 0$ (G_{2r} of any finite ring $= 0$, see [12, Theorem 7.1.12 (ii)]).

Hence, $K_{2n}\Gamma/\underline{q}(\frac{1}{p}) = 0$ and $K_{2n}\Lambda/\underline{q}(\frac{1}{p}) = 0$ in (*I*), and so, $\ker(\phi : K_{2n-1}(\Lambda) \to K_{2n-1}(\Gamma))$ and `coker` $(\varphi : K_{2n-1}(\Lambda) \to K_{2n-1}(\Gamma))$ are p-torsion.

Now, from the following commutative diagram

$$\begin{array}{ccc} K_n(\Lambda) & \xrightarrow{\varphi} & K_n(\Gamma) \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & K_n(\Sigma) & \end{array} \quad \texttt{for all n} \geq 1, \tag{III}$$

we have an exact sequence

$$0 \to \ker \varphi \to SK_n(\Lambda) \to SK_n(\Gamma) \to \texttt{Coker}\, \varphi \to \texttt{Coker}\, \alpha \to \texttt{Coker}\, \beta \to 0.$$

Put $n = 2r - 1$, we have an exact sequence

$$0 \to \ker \varphi \to SK_{2r-1}(\Lambda) \to SK_{2r-1}(\Gamma) \to \cdots . \tag{IV}$$

Now by [8] or [12, Theorem 7.1.11(iii)], $SK_{2n-1}(\Gamma)$ is a finite group. Also by [12, Theorem 7.1.3] or [5], $SK_{2r-1}(\Gamma) = 0$ iff Σ is a product of matrix algebras over fields. So, $SK_{2r-1}(\Gamma) = 0$ and $SK_{2r-1}(\Lambda) \simeq \ker \varphi$ is a finite p-group. (Note also that Γ is a regular ring, and so $G_n(\Gamma) = K_n(\Gamma)$). □

Remark 2. Note that for a finite p-group G, p an odd prime, $\widehat{\mathbb{Q}}_p G$ is indeed a product of matrix over fields, and so, $SK_{2n-1}(\widehat{\mathbb{Q}}_p G)$ is a finite p-group for all $n \geq 1$.

2. HIGHER K-THEORY OF TWISTED POLYNOMIAL RINGS $\Lambda_\alpha[t]$, Λ a p-ADIC ORDER

2.1.

Let A be a ring with identity, $\alpha : A \to A$ an automorphism of A, $A_\alpha[t]$ the α-twisted polynomial ring. We define

$$NK_n(A, \alpha) := \ker \left(K_n(A_\alpha[t]) \overset{\epsilon^*}{\to} K_n(A) \right),$$

where ϵ^* is induced by the augmentation $\epsilon : A_\alpha[t] \to A;\ t \to 0$.

Our aim is to prove the following theorem.

Theorem 2. *Let F be a p-adic field with ring of integers R, Λ any R-order in a semisimple F-algebra Σ, $\alpha : \Lambda \to \Lambda$ an automorphism of Λ. Then $NK_n(\Lambda, \alpha)$ is a p-torsion group. Hence, for any finite group G, $NK_n(RG, \alpha)$ is p-torsion.*

Proof. By adapting the proof in [12, Theorem 7.5.2] or [14] to the p-adic situation, one can see that there exists an R-order $\Gamma \subset \Sigma$ such that $\Lambda \subset \Gamma$, Γ is α-invariant and Γ is right regular and hereditary. Note that $\Sigma = \Lambda\left(\frac{1}{p}\right)$, and so $\Lambda \subset \Gamma \subset \Lambda\left(\frac{1}{p}\right)$ and $\Lambda\left(\frac{1}{p}\right) = \Gamma\left(\frac{1}{p}\right)$. Hence, there exists a positive integer s such that $\underline{q} = p^s\Gamma$ is an ideal of both Λ and Γ. Hence, we have Cartesian squares

$$(I)\quad \begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Lambda/\underline{q} & \longrightarrow & \Gamma/\underline{q} \end{array} \qquad \begin{array}{ccc} \Lambda_\alpha[t] & \longrightarrow & \Gamma_\alpha[t] \\ \downarrow & & \downarrow \\ (\Lambda/\underline{q})_\alpha[t] & \longrightarrow & (\Gamma/\underline{q})_\alpha[t] \end{array} \quad (II)$$

So, by [1], we obtain from (I) an $M-V$ sequence

$$\to K_{n+1}\left(\Gamma/\underline{q}\right)\left(\frac{1}{p}\right)\to K_n(\Lambda)\left(\frac{1}{p}\right)\to K_n(\Gamma)\left(\frac{1}{p}\right)\oplus K_n\left(\Lambda/\underline{q}\right)\left(\frac{1}{p}\right)\to$$
$$\to K_{n+1}\left(\Gamma/\underline{q}\right)\left(\frac{1}{p}\right)\to \tag{III}$$

from (II) an $M-V$ sequence

$$\to K_{n+1}\left(\Gamma/\underline{q}\right)_{\alpha}[t]\left(\frac{1}{p}\right)\to K_n(\Lambda_{\alpha}[t])\left(\frac{1}{p}\right)$$
$$\to K_n(\Gamma_{\alpha}[t])\left(\frac{1}{p}\right)\oplus K_n\left(\left(\Lambda/\underline{q}\right)_{\alpha}[t]\right)\left(\frac{1}{p}\right)\to K_{n+1}\left(\left(\Gamma/\underline{q}\right)_{\alpha}[t]\right)\left(\frac{1}{p}\right)\to . \tag{IV}$$

Mapping (*IV*) to (*III*) via the maps induced by augmentations, and taking kernels; we have an $M-V$ sequence

$$\to NK_{n+1}\left(\Gamma/\underline{q},\alpha\right)\left(\frac{1}{p}\right)\to NK_n(\Lambda,\alpha)\left(\frac{1}{p}\right)$$
$$\to NK_n\left(\Lambda/\underline{q},\alpha\right)\left(\frac{1}{p}\right)\oplus NK_n(\Gamma,\alpha)\left(\frac{1}{p}\right)\to NK_n\left(\Gamma/\underline{q},\alpha\right)\left(\frac{1}{p}\right)$$
$$\to NK_{n-1}(\Lambda,\alpha)\left(\frac{1}{p}\right)\to \cdot \tag{V}$$

Now, by [2, 3] $\Gamma_{\alpha}[t]$ is regular since Γ is. So $NK_n(\Gamma,\alpha)=0$ since $G_n(\Gamma_{\alpha}[t])\simeq G_n(\Gamma)\simeq K_n(\Gamma)$. (See [12] Theorem 7.5.3(i)). Now $\Lambda/\underline{q}$ and $\Gamma/\underline{q}$ are $=\mathbb{Z}/p^s$-algebra and finite. □

Claim. If A is a finite ring that is also a $\mathbb{Z}/p^s$-algebra, then $NK_n(A,\alpha)$ is p-torsion.

Proof. The Jacobson's radical (rad A) of A is nilpotent and by [18, Corollary 5.4] the relative groups $K_n(A,\text{rad}\,A)$ are p-torsion for all $n\geq 0$. So, from the relative sequence tensored with $\mathbb{Z}\left(\frac{1}{p}\right)$ we have that

$$K_n(A)\left(\frac{1}{p}\right)\simeq K_n(A/\text{rad}\,A)\left(\frac{1}{p}\right). \tag{VI}$$

Similarly $A_{\alpha}[t]/(\text{rad}\,A)_{\alpha}[t]\simeq(A/\text{rad}\,A)_{\alpha}[t]$ (see [2, 3]), and so, we have

$$K_n(A_{\alpha}[t])\left(\frac{1}{p}\right)\simeq K_n(A/\text{rad}\,A)_{\alpha}[t]\left(\frac{1}{p}\right). \tag{VII}$$

Since $A/\text{rad}\,A$ is regular, we have

$$K_n(A/\text{rad}\,A)_{\alpha}[t]\left(\frac{1}{p}\right)\simeq K_n(A/\text{rad}\,A)\left(\frac{1}{p}\right). \tag{VIII}$$

by [12, Theorem 7.5.3 (i)] or [14]. Hence, from (VI), (VII), (VIII), we have

$$K_n(A_\alpha[t])\left(\frac{1}{p}\right) \simeq K_n(A)\left(\frac{1}{p}\right). \tag{IX}$$

From the exact sequence

$$0 \to NK_n(A, \alpha) \to K_n(A_\alpha[t]) \to K_n(A) \to 0$$

and also from (IX), we have $NK_n(A, \alpha)\left(\frac{1}{p}\right) = 0$; i.e., $NK_n(A, \alpha)$ is p-torsion.

Since the claim applies to $A = \Lambda/\underline{q}$ and $A = \Gamma/\underline{q}$, it follows from (V) that $NK_n(\Lambda, \alpha)\left(\frac{1}{p}\right) = 0$, that is, $NK_n(\Lambda, \alpha)$ is p-torsion.

The last statement follows by putting $\Lambda = RG$, G a finite group. □

3. HIGHER K-THEORY OF TWISTED LAURENT SERIES RINGS $\Lambda_\alpha[T]$ AND $\Sigma_\alpha[T]$

3.1.

Let F be a p-adic field with ring of integers R; Λ any R-order in a semisimple F-algebra Σ, α and R-automorphism of Λ, which as earlier observed extends to an F-automorphism of $\Sigma = \Lambda\left(\frac{1}{p}\right)$. Let Γ be an α-invariant R-order containing Λ. This exists by an adaptation of the proof in [12, Theorem 7.5.2].

3.2.

Also recall that if $\Lambda = RG$ in 3.1 where G is a finite group, such that $\alpha(G) = G$, then $\Lambda_\alpha[T]$ is the group ring RV where V is a virtually infinite cyclic group of the form $V = G \rtimes_\alpha T$, a semidirect product where α is an automorphism of G and the action of the infinite cyclic group $T = \langle t \rangle$ on G is given by $\alpha(g) = tgt^{-1}$ for all $g \in G$.

The aim of this section is to prove the following theorem.

Theorem 3. *Let F be a p-adic field with ring of integers R, Λ any R-order in a semisimple F-algebra Σ, α an R-automorphism of Λ. Denote the extension of α to Σ also by α. Let $\Lambda_\alpha[T]$ (resp., $\Sigma_\alpha[T]$) be the α-twisted Laurent series ring over Λ (resp., Σ). Then for all $n \geq 2$, there exist isomorphisms*

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T])$$

Hence if $V = G \rtimes_\alpha T$ (in the notation of 3.2) then

$$\mathbb{Q} \otimes K_n(RV) \simeq \mathbb{Q} \otimes G_n(RV) \simeq \mathbb{Q} \otimes K_n(FV), \qquad \text{for all } n \geq 2.$$

The proof of Theorem 3 will follow from Theorems 4, 5, and 7 below.

Theorem 4. *Let F be a p-adic field with ring of integers R, Γ an α-invariant R-order containing an arbitrary R-order Λ in a semisimple F-algebra Σ, $\alpha : \Lambda \to \Lambda$ an*

R-automorphism and $\Lambda_\alpha[T]$ *(resp.,* $\Gamma_\alpha[T]$*) the* α*-twisted Laurent series rings over* Λ *(resp.,* Γ*). Let* $\varphi_n : G_n(\Gamma_\alpha[T]) \to G_n(\Lambda_\alpha[T])$ *be the homomorphism induced by restriction of scalars. Then for all* $n \geq 2$, φ_n *has finite kernel and cokernel and hence induces an isomorphism*

$$\mathbb{Q} \otimes G_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]).$$

Proof. First note that if the exact functor $\mathcal{M}(\Gamma) \to \mathcal{M}(\Lambda)$ is given by restriction of scalars, then as in the proof of [9, Theorem 1.8 (iii)], the induced group homomorphism $G_n(\Gamma) \to G_n(\Lambda)$ has finite kernel and cokernel. □

Now from [12, Theorem 7.5.3(b)] or [14], we have the following horizontal long exact sequences and hence a commutative diagram

$$\begin{array}{ccccccccccc} \longrightarrow G_n(\Gamma) & \xrightarrow{1-\alpha_*} & G_n(\Gamma) & \longrightarrow & G_n(\Gamma_\alpha[T]) & \longrightarrow & G_{n-1}(\Gamma) & \xrightarrow{1-\alpha_*} & G_{n-1}(\Gamma) \longrightarrow & & \\ \downarrow & & \downarrow \delta_n & & \downarrow \varphi_n & & \downarrow \delta_{n-1} & & \downarrow & & [I] \\ \to G_n(\Lambda) & \xrightarrow{1-\alpha_*} & G_n(\Lambda) & \longrightarrow & G_n(\Lambda_\alpha[T]) & \longrightarrow & G_{n-1}(\Lambda) & \xrightarrow{1-\alpha_*} & G_{n-1}(\Lambda) \longrightarrow & \cdot & \end{array}$$

We now want to prove the following result from which Theorem 4 follows.

Theorem 5. *In the commutative diagram* (*I*) *above, Ker* φ_n *and Coker* φ_n *are finite groups.*

The proof of Theorem 5 above will make use of the following Lemma 6 (see [16, p. 44]).

Lemma 6. *Given an Abelian category* $\mathcal{M}$ *and a Serre subcategory* $\mathcal{N}$*, there exists an exact functor from* $\mathcal{M}$ *to the quotient Abelian category* $\mathcal{M}/\mathcal{N}$ *whose kernel is* $\mathcal{N}$.

Proof of Theorem 5. We first recall the five lemma which says that if we have a diagram of Abelian groups

$$\begin{array}{ccccccccc} 0 & & 0 & & 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & & * & & \% & & * & & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & & * & & \% & & * & & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

with exact rows and columns and all the groups at positions marked $*$ are equal to zero, then so are the groups at position marked %. We need an analogous statement that if the groups at positions marked $*$ are finite, so are the groups at position marked %.

Lemma 6 makes it possible for us to make such a statement if we regard $\mathscr{M}$ as the category of the Abelian groups and $\mathscr{N}$ as the Serre subcategory of finite Abelian groups. This is because the five lemma holds in any Abelian category $\mathscr{M}$ and hence holds in the Abelian category $\mathscr{M}/\mathscr{N}$. So if we apply the functor in Lemma 6 from the category of Abelian groups $\mathscr{M}$ to its quotient by the Serre subcategory of finite Abelian groups, and apply the five lemma to the result, we obtain that $\ker \varphi_n$ and $\operatorname{Coker} \varphi_n$ are finite (since we know that $\ker \delta_n$ and $\operatorname{Coker} \delta_n$ are finite). Since we can choose Γ to be hereditary, hence regular, $\Gamma_\alpha[T]$ is regular, we have

$$\mathbb{Q} \otimes K_n(\Gamma_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Lambda_\alpha[T]).$$

Theorem 7. *Let F be a p-adic field with ring of integers R. Λ, α, Γ, $\Lambda_\alpha[T]$, $\Gamma_\alpha[T]$ as in* 3.1 *and* 3.2. *Let $\varphi : \Lambda_\alpha[T] \to \Gamma_\alpha[T]$ be the map induced by the inclusion map $\Lambda \to \Gamma$. Then the induced homomorphism $\varphi : K_n(\Lambda_\alpha[T]) \to K_n(\Gamma_\alpha[T])$ has p-torsion kernel and cokernel for all $n \geq 2$. Hence, we have an isomorphism*

$$\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Gamma_\alpha[T]).$$

Proof. As observed earlier, we have an ideal $\underline{q} = p^s\Gamma$ ($s =$ a positive integer) such that $\underline{q}$ is an ideal of both Γ and Λ. Put $B = \Lambda/\underline{q}$ and $B' = \Gamma/\underline{q}$. So we have Cartesian squares

$$(I)\quad \begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \qquad \begin{array}{ccc} \Lambda_\alpha([T]) & \longrightarrow & \Gamma_\alpha([T]) \\ \downarrow & & \downarrow \\ B_\alpha([T]) & \longrightarrow & B'_\alpha([T]) \end{array} \quad (II).$$

By [1], we have long associated $M - V$ sequences. For (*II*), we have the exact $M - V$ sequence

$$\begin{aligned} &\to K_{n+1}(B'_\alpha([T]))\left(\frac{1}{p}\right) \to K_n(\Lambda_\alpha([T]))\left(\frac{1}{p}\right) \\ &\to K_n(\Gamma_\alpha([T]))\left(\frac{1}{p}\right) \oplus K_n(B_\alpha([T]))\left(\frac{1}{p}\right) \to K_n(B'_\alpha([T]))\left(\frac{1}{p}\right) \to \cdots. \end{aligned} \tag{III}$$

If we write A for $B_\alpha[T]$ or $B'_\alpha([T])$, and $\operatorname{rad} A$ for the Jacobson's radical of A, we have from [18] that $K_n(A, \operatorname{rad} A)$ is p-torsion since p^s annihilates A. So, from the relative sequence

$$\cdots \to K_n(A, \operatorname{rad} A) \to K_n(A) \to K_n(A/\operatorname{rad} A)$$

tensored with $\mathbb{Z}(\frac{1}{p})$, we have that $K_n(A)(\frac{1}{p}) \simeq K_n(A/\operatorname{rad} A)(\frac{1}{p})$.

Claim. $K_n(A)\left(\frac{1}{p}\right) \simeq K_n(A/\mathrm{rad}\,A)\left(\frac{1}{p}\right)$ is torsion. Hence $\mathbb{Q} \otimes K_n(A)\left(\frac{1}{p}\right) = 0$.

Proof. Note that $(A/\mathrm{rad}\,A) \simeq (A'/\mathrm{rad}\,A')_\alpha[T]$ is a regular ring (see [3]) where $A'/\mathrm{rad}\,A'$ is a finite semisimple ring which is a finite direct product of matrix algenras over finite fields. Hence $K_n((A'/\mathrm{rad}\,A')_\alpha[T])$ is a finite direct sum of K-groups of the form $K_n((F_i)_\alpha[T])$, where F_i is a finite field. Also, $(F_i)_\alpha[T]$ is a regular ring, and so, $K_n((F_i)_\alpha[T]) \simeq G_n((F_i)_\alpha[T])$.

Now for each F_i there exists (by [12, Theorem 7.5.3 (ii)] or [14]) a natural long exact sequence

$$\cdots \to G_n(F_i) \to G_n(F_i) \to G_n(F_i)_\alpha[T] \to G_{n-1}(F_i) \to G_{n-1}(F_i) \to, \qquad \text{(IV)}$$

where each $G_n(F_i) \simeq K_n(F_i)$ is a finite Abelian group for $n \geq 2$ by [12, Theorem 7.1.12]. So, from (*IV*) above, $G_n((F_i)_\alpha[T])$ is finite for all $n \geq 2$, i.e., $K_n((F_i)_\alpha[T]) \simeq G_n((F_i)_\alpha[T])$ is a finite Abelian group. Hence $K_n(A'/\mathrm{rad}\,A')_\alpha[T]$ as a direct sum of Abelian groups of the form $K_n((F_i)_\alpha[T])$ is a finite group. Hence $K_n(A'/\mathrm{rad}\,A')_\alpha[T](\frac{1}{p})$ is torsion. So, $K_n(A)(\frac{1}{p}) \simeq K_n(A/\mathrm{rad}\,A)(\frac{1}{p}) \simeq K_n(A'/\mathrm{rad}\,A')_\alpha[T](\frac{1}{p})$ is torsion and $\mathbb{Q} \otimes K_n(A)\left(\frac{1}{p}\right) = 0$.

Conclusion of the Proof of Theorem 7

It follows from the claim above that if $A = B_\alpha[T]$ or $B'_\alpha[T]$, then $\mathbb{Q} \otimes K_n(A)\left(\frac{1}{p}\right) = 0$. Hence, by tensoring the exact sequence (*III*) with $\mathbb{Q}$, we obtain an isomorphism $\mathbb{Q} \otimes K_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Gamma_\alpha[T])$.

Theorem 8. *Let F be a p-adic field with ring of integers R, Λ, Σ, α, T as in Theorem 3. Then for all $n \geq 2$, the maps $\rho_n : G_n(\Lambda_\alpha[T]) \to G_n(\Sigma_\alpha[T]) \simeq K_n(\Sigma_\alpha[T])$ induced by the canonical map $\Lambda_\alpha[T] \to \Sigma_\alpha[T]$ have finite kernel and torsion cokernel. Hence,*

$$\mathbb{Q} \otimes G_n(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n(\Sigma_\alpha[T]) \simeq \mathbb{Q} \otimes K_n(\Sigma_\alpha[T]).$$

Proof. First note that the inclusion map $\Lambda \to \Sigma$ induces a group homomorphism

$$\varphi_n : G_n(\Lambda) \to G_n(\Sigma) \simeq K_n(\Sigma).$$

(Note that $G_n(\Sigma) \simeq K_n(\Sigma)$ since Σ is regular). Now by [12, Theorem 7.5.3(b)], we have the following horizontal exact sequences and hence a commutative diagram:

$$\begin{array}{ccccccccc}
\longrightarrow G_n(\Lambda) & \xrightarrow{1-\alpha_*} & G_n(\Lambda) & \longrightarrow & G_n(\Lambda_\alpha[T]) & \longrightarrow & G_{n-1}(\Lambda) & \longrightarrow & G_{n-1}(\Lambda) \longrightarrow \\
\downarrow \varphi_n & & \downarrow \varphi_n & & \downarrow \delta_n & & \downarrow \varphi_{n-1} & & \downarrow \varphi_{n-1} \\
\longrightarrow G_n(\Sigma) & \xrightarrow{1-\alpha_*} & G_n(\Sigma) & \longrightarrow & G_n(\Sigma_\alpha[T]) & \longrightarrow & G_{n-1}(\Sigma) & \longrightarrow & G_{n-1}(\Sigma) \longrightarrow \cdot
\end{array}$$

Also by [9, Theorem 1.8(ii)], each φ_n has finite kernel and cokernel for $n \geq 2$. φ_n has finite kernel and cokernel for $n \geq 2$. Hence,

$$\mathbb{Q} \otimes G_n\,(\Lambda_\alpha[T]) \simeq \mathbb{Q} \otimes G_n\,(\Sigma_\alpha[T]) \simeq \mathbb{Q} \otimes K_n\,(\Sigma_\alpha[T])\,.$$

(Note that $G_n((\Sigma_\alpha[T]) \simeq K_n\,(\Sigma_\alpha[T])$ since $\Sigma_\alpha[T]$ is regular.)

We close this section with the following result.

Theorem 9. *Let F be a p-adic field with ring of integers R, Λ, α, Σ, Γ, $\Lambda_\alpha[T]$, $\Gamma_\alpha[T]$, as in Theorem 4. Then $NK_n(\Lambda_\alpha[T])$ is p-torsion for all $n \geq 1$.*

Proof. As before, we have an ideal $\underline{q} = p^s\Gamma$ of both Λ and Γ. Put $B = \Lambda/\underline{q}$ and $B' = \Gamma/\underline{q}$, and so we have the Cartesian square

$$\begin{array}{ccc}
\Lambda_\alpha[T] & \longrightarrow & \Gamma_\alpha[T] \\
\downarrow & & \downarrow \\
B_\alpha[T] & \longrightarrow & B'_\alpha[T]
\end{array}\;\cdot$$

Since p^s annihilates B and B' for some s, it also annihilates $B_\alpha[T]$ and $B'_\alpha[T]$ since for $A = B$ or B', $A_\alpha[T]$ is a direct limit of free $A_\alpha[t]$-modules $A_\alpha[t]t^{-1}$. So, $NK_n\,(A_\alpha[T])$ is p-torsion using a similar argument as before. So in the exact sequence

$$\cdots \to NK_{n+1}\,(B'_\alpha([T]))\left(\frac{1}{p}\right) \to NK_n\,(\Lambda_\alpha([T]))\left(\frac{1}{p}\right)$$

$$\to NK_n\,(B_\alpha([T]))\left(\frac{1}{p}\right) \oplus NK_n\,(\Gamma_\alpha([T]))\left(\frac{1}{p}\right) \to NK_n\,(B'_\alpha([T]))\left(\frac{1}{p}\right) \to \cdots \quad \text{(I)}$$

we have $NK_n\,(\Gamma_\alpha([T])) = 0$ since $\Gamma_\alpha([T]$ is regular, $NK_n\,(B_\alpha([T]))$, $NK_n\,(B'_\alpha([T]))$ are p-torsion and so from (*I*) we have that $NK_n\,(\Lambda_\alpha([T]))\left(\frac{1}{p}\right) = 0$. So $NK_n\,(\Lambda_\alpha([T]))$ is p-torsion. In particular $NK_n\,(RV)$ is p-torsion.

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