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*Chapter 2*

# HIGHER ALGEBRAIC K – THEORY OF G – REPRESENTATIONS FOR THE ACTIONS OF FINITE AND ALGEBRAIC GROUPS G

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## INTRODUCTION

The subject of K –theory of grouprings RG, (R a commutative ring with identity, G a group) was recognized since its classical days as belonging to Group representation theory.

Indeed, when R is a field F, say of characteristic zero, G a finite group, $\mathcal{P}$ (F) the category of finite dimensional vector spaces over F, and $\mathcal{P}(F)_G$ the category of representations of G in $\mathcal{P}$ (F), then the Grothendieck group of $\mathcal{P}(F)_G$, $K_0(\mathcal{P}(F)_G)$ coincides with the Abelian group of generalized characters x: G $\rightarrow$ F. ( See [6], [39]). If we denote by $\mathcal{M}$(F) the category of finitely generated F – modules, then $\mathcal{P}$ (F) = $\mathcal{M}$ (F) since a vector space, by definition, is just a module over F and a finite dimensional vector space over field F is just a finitely generated F – module. Hence, $K_0(\mathcal{P}(F)_G) \simeq K_0(\mathcal{M}(F)_G)$. Now $\mathcal{P}(F)_G = \mathcal{M}(F)_G$ can be identified with the category $\mathcal{M}(FG)$ of finitely generated FG – modules and so $K_0(\mathcal{P}(F)_G) \simeq K_0(\mathcal{M}(F)_G) \simeq K_0(\mathcal{M}(FG))$, where $K_0(\mathcal{M}(FG))$ is usually denoted by $G_0(FG)$, a K – group of the group algebra FG.

Now let G be an algebraic group over a field F, H a subgroup of G, and G/H a homogeneous G – space which is usually a quasi – projective variety. If X is any G – space ( a G – variety or G – scheme) and we denote by $\mathcal{VB}_G(X)$ the categories of (algebraic) G – vector bundles on X, then there is an equivalence of categories between the category $\mathcal{P}(F)_H$ of representations of H in $\mathcal{P}$ (F) and the category $\mathcal{VB}_G(G/H)$ of algebraic G –

vector bundles on G/H. (See [55]).
Hence $K_0(\mathcal{M}(F)_H) = K_0(\mathcal{P}(F)_H) = K_0(\mathcal{VB}_G(G/H))$. We denote the group $K_0(\mathcal{VB}_G(G/H))$ by $K_0^G(G/H)$. For a G – scheme X, we denote $K_0(\mathcal{VB}_G(X))$ by $K_0^G(X)$

Note that for any scheme X, (no G – action), the category $\mathcal{P}$(X) of locally free sheaves of $O_X$ -modules, and $\mathcal{VB}$(X) the category of algebraic vector bundles on X are equivalent and $\mathcal{P}$(X), $\mathcal{VB}$(X) and $\mathcal{P}$(F) are examples of (ordinary) exact categories. More over, if X is a G – space, G an algebraic group, the category of G – modules that are locally free $O_X$ -modules (see [14] [50]), then $\mathcal{P}$ (G, X) is equivalent to $\mathcal{VB}_G(X)$ and $\mathcal{P}$ (G, X), $\mathcal{VB}_G(X)$ and $\mathcal{P}(F)_G$ are examples of equivariant exact categories. Our aim in this article is to present computations of $K_n$ of the various equivariant exact categories specified in chapter I of this article.

Now observe that any vector space V $\in$ $\mathcal{P}$(F) is free and hence projective and so, for any ring R with identity, the category $\mathcal{P}$ (R)of finitely generated projective R – modules is a natural generalization of $\mathcal{P}$(F) since $\mathcal{P}$(R) = $\mathcal{P}$(F) when R = F. If we write $\mathcal{M}$(R) for the category of finitely generated R – modules, then the categories $\mathcal{P}(R)_G$, (resp $\mathcal{M}(R)_G$) makes sense. Indeed the category $\mathcal{C}_G$ of representations of G in arbitrary category $\mathcal{C}$ makes sense. (see [39]). Now, when R is commutative and G is finite $\mathcal{P}(R)_G$ can be identified with the category $\mathcal{P}_R(RG)$ of RG – lattices i.e. RG – modules that are finitely generated and R-projective. Again $\mathcal{P}$(R), $\mathcal{M}$(R) are examples of (ordinary) exact categories yielding (ordinary) higher K – groups $K_n(\mathcal{M}(R)) := G_n(R)$ and $K_n(\mathcal{P}(R)) :=$ $K_n(R)$ while $\mathcal{M}(R)_G$, $\mathcal{P}(R)_G$ are examples of equivariant exact categories yielding higher K – groups $K_n(\mathcal{M}(R)_G) \simeq K_n(\mathcal{M}(RG) := G_n(RG)$ and $K_n(\mathcal{P}(R)_G) \simeq K_n(\mathcal{P}_R(RG) := G_n(R,G)$.
When R is the ring $\mathbb{Z}, \mathbb{Q}, \hat{\mathbb{Z}}_P, \hat{\mathbb{Q}}_P$ or more generally the ring of integers R ( resp $\hat{\mathbb{R}}_{\underline{P}}$) in number field F (resp $\hat{F}_{\underline{P}}$ where $\underline{P}$ is a prime ideal of R), computations of $K_n(RG), G_n(RG), K_n(\hat{R}_{\underline{P}}G), G_n(\hat{R}_{\underline{P}}G)$ n ≥ 0 belong to the realm of integral representation theory and we present computations of these groups copiously in section 1 of chapter III.

Here is a brief outline of the contents of this article. As already remarked, the theme of the article is to present computations of higher K - theory of various examples of equivariant exact categories given in chapter I for the actions of finite and algebraic groups. In chapter II, we introduce higher algebraic K – theory of ordinary as well as equivariant exact categories with capious examples for the actions of finite and algebraic groups. In section 2 of chapter II, we discuss induction techniques for higher K – theory by realizing equivariant higher K – theory as Mackey functors yielding some explicit results on higher K – theory of grouprings. Chapter III is devoted to presenting explicit computations of higher K – theory (including profinite higher K – theory) of the various examples of equivariant exact categories encountered in chapter I and II for the action of finite and algebraic groups.

Time and space prevented us from discussing actions of other groups e.g. profinite groups, compact Lie groups as well as group actions on objects of other categories e.g. Waldhausen and symmetric monoidal categories. However, interested readers could see [39] for more information.

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# CHAPTER I. EQUIVARIANT EXACT CATEGORIES

We shall in this article focus on exact categories which already yield many examples, even though analogous constructions and results can be obtained for other categories e.g. symmetric monoidal categories and Waldhausen categories (See [39].)

## Section 1. Exact Categories and Some Relevant Examples

***1.1. Definition***

An exact category is a small additive category $\mathcal{C}$ embeddable as a full subcategory of an Abelian category $\mathcal{A}$ such that $\mathcal{C}$ is equipped with a class $\mathcal{E}$ of short exact sequences 0 $\to$M´$\to$M$\to$M´´$\to$ 0 (1) satisfying

(1) $\mathcal{E}$ is the class of sequences (1) in $\mathcal{C}$ that are exact in $\mathcal{A}$.
(2) $\mathcal{C}$ is closed under extensions in $\mathcal{A}$. that is, if (1) is an exact sequence in A, and M´, M´´$\in$ ob($\mathcal{C}$), then M$\in$ob($\mathcal{C}$).

***1.2. Examples***

*1.2.1. Let A be any ring with identity*. Then $\mathcal{P}$ (A), the category of finitely generated projective A-modules is an exact category. We shall be interested in A = RG where R is a Dedekind domain with quotient field F (e.g. R = ring of integers in a number field or p-adic field). More generally, A could be an R- order in a semi – simple F – algebra $\Sigma$ (see [6]. [39])

If A = RG, we shall also be interested in the group $(RG)_\alpha[T] = RV$ where $V = G \rtimes_\alpha T$ - G a finite group, and the action of the infinite cyclic group T = < t > on G is given by $\alpha(g) = tgt^{-1}$ for all g $\in$ G. Note that V is called a virtually infinite cyclic group. These groups are important in the context of Farrell – Jones conjecture in K-theory (see [39])

*1.2.2. Let A be a Noetherian ring*. Then $\mathcal{M}$ (A) the category of finitely generated A - modules. is an exact category.

*1.2.3. Let X be a scheme.* Recall that X is a ringed space $(X,O_X)$ where X is covered by open sets $U_i$ such that $(U_i,O_X|_{U_i})$ is an affine scheme (see [14]). Then $\mathcal{P}$(X), the category of locally free sheaves of $O_X$ -modules is exact. Note that this category is identified as the category of sheaves of sections of the algebraic vector bundles on X (where an algebraic vector bundle is a scheme morphism π: E → X together with maps E x E → E ; μ : $\mathbb{A}^1_{\mathbb{Z}} \underset{Spec(\mathbb{Z})}{\times} E \to E$, satisfying axioms similar to those of topological vector bundles, together with local triviality, that is there exists an open covering $X = \cup U_\alpha$ of X together with isomorphism. $E|U_\alpha \simeq \pi^{-1}(U\alpha) \simeq \mathbb{A}^n_{U_\alpha}$ See [14]. As such, $\mathcal{P}$ (X) is equivalent to the category $\mathcal{VB}$(X) of algebraic vector bundles on X.

Note that if X = Spec (R) where R is a commutative ring with identity, then we have an equivalence of categories $\mathcal{P}(X) \simeq \mathcal{P}(X)$: E → Γ (X, E) = {R-module of global sections} with inverse equivalence $\mathcal{P}$(R) → $\mathcal{P}$(X) given by $P \to \overline{P} : U \to O_X(U) \otimes_R P$. (See [14] or [36])

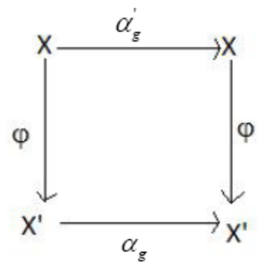
*1.2.4. Let X be a Neoetherian scheme*. Then $\mathcal{M}$(X), the category of coherent sheaves of $O_X$ -modules is an exact category. Note also that if X = Spec (R) where R is a commutative ring with identity then we have an equivalence of categories $\mathcal{M}(X) \simeq \mathcal{M}(R)$ (see [14] or [65])

*1.2.5. Let X be a compact space, F = $\mathbb{R}$ or $\mathbb{C}$.* Then the category of finite dimensional topological F-vector bundles over X is an exact category denoted by $\mathcal{VB}_F(X)$ ( See [36]).

## Section 2. Equivariant Exact Categories for the Action of Finite Groups

### *2.1. Category of G- Representations*

*2.1.1. Let G be a finite group, C an exact category*. A representation of G in $\mathbb{C}$ is a pair (X,α), X ∈ ob$\mathbb{C}$ and α : G → Aut(X) where Aut(X) is the group of $\mathbb{C}$- automorphisms of X. The representations of G in $\mathbb{C}$ also form a category which we denote by $\mathbb{C}_G$ where for (X,α), (X´,α´) ∈ ob $\mathbb{C}_G$ , a $\mathbb{C}_G$ – morphism (X,α)→(X´,α´) consists of a $\mathbb{C}$-morphism $\varphi$ : X → X´ such that for each g ∈G, the diagram

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### *2.2. Examples*

*2.2.1. Let F be a field and $\mathcal{P}$(F) the category of finite dimensional vector spaces over F – an exact category. If* $\alpha$ : G→ Aut(V) (V∈$\mathcal{P}$(F)) is a representation of G in $\mathcal{P}$(F), then there is a unique way of extending α to a representation $\bar{\alpha}$ of FG with representation space V ( that is $\bar{\alpha}\left(\sum a_g g\right)=\sum a_g \alpha(g)$ ) where $a_g \in$ F. Conversely, every representation of FG, when restricted to G yields a representation of G. Hence, there is a one – one correspondence between representations of G in $\mathcal{P}$ (F) and the category $\mathcal{M}$(FG) of finitely generated FG-modules. This in fact presents the first connection between K-theory of $\mathcal{P}(\mathrm{F})_G$ and K-theory of the group algebra FG as we shall see in II, section 1. More precisely $\mathrm{K}_0(\mathcal{P}(\mathrm{F})_G \simeq \mathrm{K}_0(\mathcal{M}(\mathrm{FG})) \simeq \mathrm{G}_0(\mathrm{FG})$ can be identified with the abelian group of generalized F-characters of G (See [6]). We shall sometimes just refer to a representation of G as a G – module.

*2.2.2. Let R be a commutative ring with identity and G a finite group*. The category $\mathcal{P}_R(RG)$ of $RG$ -lattices (i.e. $RG$ -modules that are finitely generated and projective as R-modules) is the category of representations of G in P(R) i.e. $\mathcal{P}_R(RG) \simeq \mathcal{P}(R)_G$ . Note that when R is a field F, we recover $\mathcal{P}(F)_G \simeq \mathcal{M}(FG)$. We shall denote $K_0(\mathcal{P}_R(RG))$ by $G_0(R,G)$ .

### *2.3. G-Representations as Functor Categories*

In this sub-section, we indicate how to realize the category $\mathcal{C}_G$ of G-representations as functor categories which will be useful in various other contexts.

*2.3.1. Let G be a finite group*. We can regard G as a category $\underline{G}$ with one object and whose morphisms are elements of G.

Let [ $\underline{G}$ ,$\mathcal{C}$] be the category of covariant functors from $\underline{G}$ to $\mathcal{C}$. Then [G,$\mathcal{C}$ ] is an exact category (see [39]).

*2.3.2. As a generalization of the situation in 2.3.1, let S be a (finite) G-set*. We can associate to S a category $\underline{S}$ as follows: ob( $\underline{S}$ ) = elements of S.

$\mathrm{Hom}_{\underline{S}}(s,t) = \{(g,s) | g \in S, gs = t\}$. Composition of morphisms is defined by $(h,t)\circ(g,s) = (hg,s)$ and the identity morphism S → S is (e, s) where e is the identity of

G. Call $\underline{S}$ the translation category of S. Note that S can be written as disjoint union of G-sets of the form G/H where H is a sub group of G. See [39] or [27] or [8])

*2.3.3. Let* $\underline{S}$ *be as in 2.3.2,* $\mathcal{C}$ *an exact category,* $[\underline{S}, \mathcal{C}]$ *the category of covariant functors* $\zeta : \underline{S} \to \mathcal{C}$. Define a sequence $0 \to \zeta' \to \zeta \to \zeta'' \to 0$ to be exact in $[\underline{S}, \mathcal{C}]$ if for each $s \in S$, $0 \to \zeta'(s) \to \zeta(s) \to \zeta''(s) \to 0$ is exact in $\mathcal{C}$.

One easily checks that $[\underline{S}, \mathcal{C}]$ is an exact category (see [39]).

- If S = G/G, then $[\underline{G/G}, \mathcal{C}] \simeq \mathcal{C}_G$
- Hence if F is field $\left[\underline{G/G}, \mathcal{P}(F)\right] \simeq \mathcal{P}(F)_G \simeq \mathcal{M}(FG)$
- If R is commutative ring with identity then $\left[\underline{G/G}, \mathcal{M}(R)\right] \simeq \mathcal{M}(R)_G \simeq \mathcal{M}(RG)$ where $\mathcal{M}(RG)$ is the category of finitely generated RG–modules If S = G/H, then $\left[G/H, \mathcal{M}(R)\right] \simeq \mathcal{M}(RH)$ (see [39])
- If R is commutative ring with identity, then $\left[\underline{G/G}, \mathcal{P}(R)\right] \simeq \mathcal{P}(R)_G \simeq \mathcal{P}_R(RG)$ where $\mathcal{P}_R(RG)$ is the category of RG- lattices i.e.: RG – modules that are finitely generated and projective over R. If S = G/H, then $\left[G/H, \mathcal{P}(R)\right] \simeq \mathcal{P}(R)_H \simeq \mathcal{P}_R(RH)$ Note also that $[\underline{G/H}, \mathcal{C}] \simeq [\underline{H/H}, \mathcal{C}]$ . for any category $\mathcal{C}$ (See [39])
- It is also desirable to understand the category $\mathcal{P}(RG)$ of finitely generated projective RG-modules also as a category of representations as we have done for $\mathcal{M}(RG)$ and $\mathcal{P}_R(RG)$. We do this 2.4 below.

### *2.4. Relative G- Representations as Functor Categories*

The aim of this subsection is to realize the category $\mathcal{P}(RG)$ of finitely generated projective RG-modules as functor categories This is done by first discussing the relative version of 2.3 and interpreting this situation in terms of groupings ( see 2.4.3 below)

*2.4.1. Definition*. Let S, T be G-sets. Then the projection map $\varphi : S \times T \to S$ induces a functor $\underline{\varphi} : \underline{S} \times \underline{T} \to \underline{S}$. If $\mathcal{C}$ is an exact category, and $\zeta \in [\underline{S}, \mathcal{C}]$, we write $\zeta' = \zeta \circ \varphi : \underline{S} \times \underline{T} \to \underline{S} \to \mathcal{C}$, Then, a sequence $\zeta_1 \to \zeta_2 \to \zeta_3$ of functors in $[\underline{S}, \mathcal{C}]$ is said to be T-exact if the sequence $\zeta_1' \to \zeta_2' \to \zeta_3'$ of restricted functors $\underline{S} \times \underline{T} \to \underline{S} \to \mathcal{C}$ is split exact. We now define a new exact category ${}^T\left[\underline{S}, \mathcal{C}\right]$, as follows ob ${}^T\left[\underline{S}, \mathcal{C}\right]$ = ob $\left[\underline{S}, \mathcal{C}\right]$. Exact sequences in ${}^T\left[\underline{S}, \mathcal{C}\right]$ are T-exact sequences in $\left[\underline{S}, \mathcal{C}\right]$.

*2.4.2. Let S = G/H, H ≤ G, R a commutative ring with identity,* T a G- set. Then a sequence of functors $\zeta_1 \to \zeta_2 \to \zeta_3$ in $\left[G/H, \mathcal{M}(R)\right]$ (resp $\left[\underline{G/H}, \mathcal{P}(R)\right]$ is said to be T- exact if $\zeta_1(\mathrm{H}) \to \zeta_2(\mathrm{H}) \to \zeta_3(\mathrm{H})$ is RH′ - split exact for all H′ ≤ H such that $T^{H'} \neq \phi$ where $T^{H'}$ = {t∈T| gt = t for all g ∈H′}. Recall that $\left[\underline{G/H}, \mathcal{M}(R)\right] \simeq \mathcal{M}(RH)$ and $\left[\underline{G/H}, \mathcal{P}(R)\right] = \mathcal{P}_R(RH)$. Hence the sequence is G/H –exact (resp G/G – exact) iff the corresponding sequence of RH –modules (resp RG – modules) is split exact. If e is the trivial subgroup of G, then the sequence is G/e – exact if it is split exact as a sequence of R-modules.

So, ${}^T\left[\underline{G/H}, \mathcal{P}(R)\right]$ (resp ${}^T\left[\underline{G/H}, \mathcal{M}(R)\right]$) is the exact category of M ∈ $\mathcal{P}_R(RH)$ (resp M ∈ $\mathcal{M}(RH)$) with respect to exact sequence that split when restricted to various subgroups H′ of H such that $T^{H'} \neq \phi$.

*Definition 2.4.3.* Let S, T be G-sets , $\mathcal{C}$ an exact category. A functor $\zeta \in \left[\underline{S}, \mathcal{C}\right]$ is said to be T-projective if any T-exact sequence $\zeta_1 \to \zeta_2 \to \zeta$ is split exact . Let $\left[\underline{S}, \mathcal{C}\right]_T$ be the additive category of T-projective functors in $\left[\underline{S}, \mathcal{C}\right]$ considered as an exact category with respect to split exact sequences .

If S = G/H, $\mathcal{C} = \mathcal{P}(R)$ T= G/e then $\left[\underline{G/H}, \mathcal{P}(R)\right]_{G/e} \simeq \mathcal{P}(RH)$

## Section 3. Equvariant Exact Categories for the Actions of Algebraic Groups

In this section, we shall be concerned with actions of algebraic groups on $\mathcal{P}$(X) (resp $\mathcal{M}$(X)), the exact category of locally free (resp coherent) sheaves of $O_X$ - modules where X is a G - scheme. We shall also consider actions of G on various ramifications of $\mathcal{P}$(X) and $\mathcal{M}$(X) that we shall define below. While the meaning of group- action on objects of an exact category as discussed in section 2 is also valid here, the notations in section 2 would be rather cumbersome especially for various ramifications of the exact categories $\mathcal{P}$(X), $\mathcal{M}$(X) and so we adopt notations originally due to R- Thomason(see [72]).

### *3.1. Some Generalities on Algebraic Groups*

*3.1.1. Definition.* A (linear) algebraic group over a field F is an affine F-variety together with F- morphisms G x G → G: (x, y) → x y (multiplication); G → G : $x \to x^{-1}$ (inversion) such that the usual group axioms are satisfied and such that the unit element 1 is F – rational. We sometimes write G (F) for G and also call it an affine algebraic group. It follows from this definition that G (F′) is an algebraic group for any extension field F′ of F. We also denote G (F′) by $G \underset{F}{\times} F'$. We shall often omit the word ‘linear’ and just call G an algebraic group or call G an F-group.

Note that every algebraic group G can be embedded as a closed F-subgroup of some $GL_n(F)$

*3.1.2. Examples.*

(1) $GL_n(F) = \left\{(x, y) \in F^{n^2+1} \mid \det(x_{ij})y = 1\right\}$

(2) $SL_n(F) = \{(x_{ij}) \in GL_n(F) \mid \det(x_{ij}) - 1 = 0\}$

(3) The "additive group" $G_a$ is defined by $G_a$ (K) =K

(4) The "multiplicative group" is given by $G_m = GL_1$ i.e.

$G_m(F) = \left\{(x, y) \in F^2 \mid x.y = 1\right\}$

(5) The symplectic group $Sp_{2n}(F)$ is defined by $Sp_{2n}(F) = \{x \in GL_{2n}(F) \mid xJx^t = J$

where $J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \in GL_{2n}(F)$

(6) Let $q$ be a regular quadratic form on an n-dimensional vector space V over a field F. Then its associated bilinear form is $< x, y > = q(x + y) - q(x) - q(y)$ for $x, y \in$V.

If $\{e_1....e_n\}$ is a basis of V, the form $q$ is represented by the symmetric matrix $A = \left(\left\langle e_i, e_j \right\rangle\right)_{i.j=1...n} \in GL_n(F)$. Then the orthogonal group of $q$ is defined by $O_q(F) = \left\{x \in GL_n(F) \mid xAx^t - A = 0\right\}$ while the special orthogonal group of $q$ is defined by $SO_q(F) = \left\{x \in GL_n(F) \mid xAx^t - A = 0, \det x - 1 = 0\right\}$

*3.1.3. Note that an algebraic group is said to be connected (resp irreducible) if it is connected (resp irreducible) as a variety*. The connected component of G containing the identity is called its 1-component and denoted by $G_0$. $G_0$ has finite index in G.

For example $GL_n, SL_n, SO_q, Sp_{2n}(F), G_a, G_m$ are all connected. The 1-connected component of $O_q$ is $SO_q$

*3.1.4. An algebraic group G is said to be unipotent if any $x$ in G is unipotent*. If we regard G as a matrix group, this means that $x - 1_n$ is nilpotent i.e. the only eigenvalue of $x$ is 1.

G is said to be solvable if G is solvable as an abstract group. In particular, unipotent groups are solvable.

The radical r(G) (resp unipotent radical $r_u(G)$) of G is the unique maximal closed connected solvable (resp unipotent) normal subgroup of G. We have $r_u(G) \subset r(G)$ and

$r_u(G) = r(G)_u$. G is said to be reductive (resp semi-simple) if $r_u(G) = \{e\}$ (resp $r(G) = \{e\}$)

*3.1.5. (a) We also have the concepts of quotients*. If H is a closed subgroup of G, there is an essentially unique quotient F-variety G/H which is an algebraic group if H is normal. As a variety G/H is not necessary affine (see [64]). We next introduce the notion of homogeneous space. (b) Let G be an algebraic group. X an algebraic variety. An action of G on X is a morphism G x X $\to$ X: (g, $x$) $\to$ g x such that

(a) g (h $x$) = (g h)( $x$) for all g, h $\in$ G, $x \in$ X
(b) e $x$ = $x$ for all $x \in$ X. X, equipped with a G-action is called a G – variety or G-space. If X is a scheme, call X a G-scheme.
(c) A homogeneous space for G is a G-space X on which G acts transitively i.e. there exists only one G-orbit. In this case, all the isotropy groups (stabilizers) $G_x$, $x \in$ X are conjugate in G. If we fix a point $x_0 \in X$ and H = $G_{x_0} \le G$, we have a bijection $X \to G/H : \; gx_0 \to gH$

Thus, if H is closed subgroup of G, a quotient of G by H is a pair $(X, x_0), x_0 \in X$ with isotropy group $H = G_{x_0}$ such that the following universal property holds. For any pair $(Y, y_0)$ consisting of homogeneous space Y for G and $y_0 \in$Y, such that $G_{y_0} \supset H$, there exists a unique G-morphism $\varphi$ : X $\to$ Y such that $\varphi(x_0) = y_0$.

*Note* that $X \simeq G/H$ is usually a quasi-projective variety.

*3.1.6. (a) An algebraic group* T over F is called an F-torus if over some algebraic field extension F´ over F, $T \underset{F}{\times} F' \simeq \Pi G_m$ is a (finite) product of multiplicative groups $G_m$

In particular $T \times_F F_s = \Pi G_m$ for a separable closure $F_s$ of F. We shall also sometimes write $F_{sep}$ for separable closure of F. (b) An F-torus T is said to be split if T $\simeq \Pi G_m$ over F. Say that T is anisotropic if it does not contain any split subtorus (see [64]) For example $T(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in SL_2(\mathbb{R}) \mid x^2 + y^2 = 1 \right\}$ is an anisotropie torus since it is compact and hence cannot be isomorphic to $\mathbb{R}^*$ (c) Let T be an F-torus, and $F_s$ a separable closure of F. Then the Abelian group of characters $X(T) = Hom_{F_s}(T, G_m)$ is a module over the Galois group $\Gamma = Gal(F_s / F)$.

*Note:*

(1) X(T) is free Abelian group and T splits if and only if $\Gamma$ operates trivially on X(T)
(2) T is a anisotropic iff $X(T)^{\Gamma} = \{0\}$

(3) There exists a unique maximal anisotropic F-sub torus $T_a$ of $T$ and $T_a.T_s = T$ and $T_a \cap T_s$ is finite (see [64]) where $T_s$ is a unique maximal split K-subtorus of T

(d) 1) A torus is connected and Abelian and hence solvable and all its elements are semi simple. Conversely if G is connected Abelian algebraic group, all of whose elements are semi simple, then G is torus

2) The set of closed tori in G, ordered by inclusion, has a maximal element – a maximal torus. The maximal tori are all conjugate and lie in $G_0$. Hence we may assume that G is connected

(e) If $G = GL_n$, the F-subgroup T of diagonal matrices is a maximal torus in G

*3.1.7. Let G be a connected F-group.* A maximal connected solvable F-subgroup of G is called a Borel subgroup of G and usually denoted by B

An F-subgroup of G is called a parabolic subgroup if it contains a Borel subgroup.

It is usual to denote a parabolic subgroup by P.

If $G = GL_n$, then the F-subgroup B of upper triangular matrices is a Borel subgroup of G

*Note:*

(1) Every maximal torus is connected and solvable and hence is contained in some Borel subgroup.

(2) All Borel subgroups in G are conjugate over $\overline{F}$. Every element of G is contained in such a group.

(3) Let P be a closed F-subgroup of G. The quotient G/P is projective iff P is parabolic. If P is a parabolic subgroup of G, then it is connected and equal to its own normalizer in G (i.e. $P = N_G(P)$)

(4) If P,Q are two parabolic subgroups of G, and if they are conjugate then P = Q

*3.1.8. Let G, G´ be F-groups.* An F-morphism α: G´→G is called an isogeny if kerα is finite and α is surjective over $\overline{F}$ (This means that G´,G are of the same dimension). An isogeny α is central if kerα ⊂ centre of G´. Two F-groups G, H are said to be strictly isogeneous if there exists an F-group G´ and central isogenies α: G´→G and α´: G → H

A semi-simple F-group G is simply connected if there is no proper isogeny G´ → G with a semi-simple F-group G´. G is adjoint if its centre is trivial.

*3.1.9. (a) A connected solvable F-group G* is called split if there exist a series of subgroups $G_{i+1} \subset G_i \ldots$ such that $G_i / G_{i+1}$ is isomorphic to either $G_a$ or $G_m$ for $i = 0,1,\ldots,n-1$.

- A reductive F-group G is called split if it has a maximal torus which splits over F
- A reductive group G over F is called quasi-split if it has a Borel subgroup defined over F

(b) Let G, H, be algebraic groups over F G is called a twisted form of H if $G_{sep}$ and $H_{sep}$ are isomorphic ova $F_{sep}$ where $F_{sep}$ is the separable closure of F.

***3.2. Representations of G in P (F)***

*3.2.1. Let G an F - group,* $\mathcal{P}$(F) the category of finite-dimensional vector spaces over F, $\mathcal{P}(F)_G$ or $Rep_F(G)$ the category of representations of G in $\mathcal{P}$(F). Recall from section 2 that objects of $\mathcal{P}(F)_G$ are of the form (V, α: G → Aut(V)) where V ∈ $\mathcal{P}$(F)

*3.2.2. Now, for any G-scheme* X, Let $\mathcal{VB}_G(X)$ be the category of G-equivariant (algebraic) vector bundles or X. This category is also denoted by $\mathcal{P}$(G, X) (see 3.3.2)

Let H be a closed subgroup of G and X the homogeneous space G/H.

Then we have an equivalence of categories

$$\operatorname{Re} p_F(H) \underset{res}{\overset{ind}{\rightleftarrows}} \mathcal{VB}_G(G/H)$$

where 'ind' and 'res' are defined as follows:

- res: For any vector bundle $E \xrightarrow{P} G/H,\ p^{-1}(\bar{e}) \in \operatorname{Re} p_F(H)$ ( where $\bar{e}$ =eH=H) since the stabilizer of H in G/H = $\bar{e}$ .
- ind: Let ( V,α : H → Aut(V)) ∈ $\operatorname{Re} p_F(H)$. Then, one has a vector bundle (G x V) / H→ G/H where H acts on $(G\times V)/H$ by $(g,v)h = (g\cdot h, h^{-1}v)$, see [64]. We denote (G x V)/H by $\tilde{V}$ . Here $\alpha(h^{-1})v$ .

*3.2.3. Let* $\tilde{G}$ *be a semi-simple connected and simply connected*, F-split algebraic group over a fiel F. Let $\tilde{P} \subset \tilde{G}$ be a parabolic subgroup of $\tilde{G}$ containing the torus $\tilde{T}$ . The factor variety $\mathcal{F} = \tilde{G}/\tilde{P}$ is smooth and projective (see [64], [65]). Call $\mathcal{F} = \tilde{G}/\tilde{P}$ a flag variety.

- Let $N_{\tilde{G}}(\tilde{T})$ be the normalizer of $\tilde{T}$ in $\tilde{G}$ , W: = $N_{\tilde{G}}(\tilde{T})/\tilde{T}$ the Wey1 group of $\widetilde{G}$ – a finite group. Let $W_{\tilde{P}} := \{w \in W \mid w\widetilde{P}w^{-1} = \widetilde{P}\}$. Put s(F) = $\left[W : W_{\tilde{P}}\right]$.
- Let $\tilde{Z}$ be the center of $\tilde{G}$ and $\tilde{Z}^* = Hom(\tilde{Z}, G_m)$ the group of characters of $\tilde{Z}$ . Note that $\tilde{Z}^*$ is a finite group.
- Let $\chi \in \tilde{Z}^*$ and $\operatorname{Re} p_G^{\chi}(\tilde{P})$ be the full subcategory of $\operatorname{Re} p_F(\tilde{P})$ consisting of those V ∈ $\operatorname{Re} p_F(\tilde{P})$ such that $\tilde{Z}$ acts on V by the character $\chi$. The F-group scheme $\tilde{Z}$ acts on V by the charater $\chi$ and hence on every $\tilde{V} = (\tilde{G}\times V)/\tilde{P} \in \mathcal{VB}_{\tilde{G}}(\mathcal{F})$ See 3.2.2 above.
- We write $\mathcal{VB}_{\tilde{G}}(\mathcal{F},\chi)$ for the full subcategory of $\mathcal{VB}_{\tilde{G}}(\mathcal{F})$ consisting of $\tilde{V}$ such that $\tilde{Z}$ acts on every fibre of $\tilde{V}$ by the character $\chi$

***3.3. G- Modules on G-spaces X***

*3.3.1. Let G be an F-group and X a G-scheme* with G-action on X given by $\theta : G \times X \to X$ , $(g,x) \to gx$ . Let $p_2 : G \times X \to X$ : $(g,x) \to x$ be the projection onto X, $\mathcal{M}$(X) the category of coherent $O_X$ -modules. Then a coherent G-module on X is a coherent $O_X$ -module $\mathcal{F}$ together with an isomorphism $\varphi : \theta^* \mathcal{F} \simeq p_2{}^* \mathcal{F}$ of $O_{G \times X}$ -modules on $G \times X$ .

*3.3.2. Let $\mathcal{M}$ (G, X) be the category of coherent G-modules* that are coherent as $O_X$ -modules. Then $\mathcal{M}$ (G, X) is an Abelian category (which is also exact).

Let $\mathcal{P}$ (G, X) be the subcategory of those coherent modules that are vector bundles on X (i.e. locally free sheaves of $O_X$ -modules. Then $\mathcal{P}$ (G, X) is an exact category and $P_2^*$ , $\theta^*$ : $\mathcal{P}$(X) →$\mathcal{P}$(G, X) are exact functors where $\mathcal{P}$(X) is the category of locally free sheaves of $O_X$ -modules (or equivalently vector bundles on X)

*3.3.3. We have the following elaborations on the situation in 3.3.2.* (a) Let A be a finite – dimensional separable F-algebra, G an algebraic F-group and X a G-scheme. A G-A-module over a G-scheme X is a G-module M which is also a left $A \otimes_F O_X$ -module such that $g(am) = ga \cdot gm$ for g ∈ G, m ∈ M. (b) Let $\mathcal{M}$(G, X, A) be the category whose objects are G-A-modules over X and whose morphisms are $A \otimes_F O_X$ - and G-module morphism. Then, $\mathcal{M}$ (G, X, A) is an Abelian category.

Let $\mathcal{P}$ (G, X, A) be the full subcategory of $\mathcal{M}$ (G,X,A) consisting of locally free $O_{A \otimes O_X}$ - module. Then $\mathcal{P}$ (G, X, A) is an exact category.

# CHAPTER II. HIGHER K-THEORY OF EQUIVARIANT EXACT CATEGORIES – DEFINITIONS, EXAMPLES, AND SOME RESULTS

## Section 1. Brief Review of $K_n(\mathcal{C})$, n ≥ 0, $\mathcal{C}$ an Exact Category

In this section, we provide definitions and relevant examples of higher K-theory of exact categories $\mathcal{C}$ (including equivariant exact categories), thus developing notations for later use in the envisaged results.

### *1.1. Definition of $K_n(\mathcal{C})$*

*1.1.1. Definition* Let Δ be category defined as follows: ob(Δ): = { $\underline{n}$ ={0 < 1< … < n}} $Hom_\Delta(\underline{m}, \underline{n})$ = {monotonic maps $f, \underline{m} \to \underline{n}$ i.e., $f(i) \le f(j)$ for $i < j$ }. For any category

$\mathcal{A}$, a simplical object in $\mathcal{A}$ is a contravariant functor X: $\Delta \to \mathcal{A}$. Write $X_n$ for $X(\underline{n})$ A cosimplical object in $\mathcal{A}$ is a covariant functor X: $\Delta \to \mathcal{A}$.

– Equivalently, one could define a simplical object in a category $\mathcal{A}$ as a set of objects $X_n$ (n≥0) in A and a set of morphisms $\delta_i : X_n \to X_{n+1}$ ($0 \le i \le$n) called face maps as well as a set of morphisms $s_j : X_n \to X_{n+1}$ ($0 \le j \le$n) called degeneracies satisfying certain simplical identities (see [57])
– The geometric n-simplex is the topological space $\hat{\Delta}^n = \{(x_0, x_1, ...., x_n) \in R^{n+1} \mid 0 \le x_i \le 1 \forall i \text{ and} \sum x_i = 1\}$ and A functor $\hat{\Delta} : \Delta \to$ Spaces : $\underline{n} \to \hat{\Delta}^n$ is a co-simplical space.

*1.1.2. Definition:* Let $X_*$ be a simplical set. The geometric realization of $X_*$ written $|X_*|$ is defined by $|X_*| = X \times_{\Delta} \hat{\Delta} = \bigcup_{n \ge 0} \left(X_n \times \hat{\Delta}^n\right) / \cong$ where the equivalence relations $\cong$ is generated by $(x, \varphi_*(y)) \cong (\varphi^*(x), y)$ for any x $\in X_n$ y $\in Y_m$ and $\varphi : \underline{m} \to \underline{n}$ in $\Delta$ and where $X_n \times \hat{\Delta}^n$ is given the product topology and $X_n$ is considered as a discrete space.

*1.1.3. Definition.* Now let $\mathcal{A}$ be a small category. The nerve of $\mathcal{A}$, written N$\mathcal{A}$, is the simplical set whose n-simplices are diagrams:

$A_n = \{A_0 \xrightarrow{f_1} A_1 \to ... \xrightarrow{f_n} A_n\}$ where the $A_i$ 's are $\mathcal{A}$-objects and the $f_i$ are A-morphisms. The classifying space of $\mathcal{A}$ is defined as $|N\mathcal{A}|$ and denoted by B$\mathcal{A}$.

*Remarks:* B$\mathcal{A}$ is a CW-complex whose n-cells are in one-one correspondence with the non-degenerate diagram $A_n$ above.(see [57])

*1.1.4. Definition* Now let $\mathbb{C}$ be an exact category. We form a new category Q$\mathbb{C}$ such that ob(Q$\mathbb{C}$) = ob $\mathbb{C}$ and morphisms from M to P, say is an isomorphism class of diagrams $M \xleftarrow{j} N \xrightarrow{i} P$ where $i$ an admissible monomorphism ( or inflation) and $j$ is an admissible epi morphism or deflation) in $\mathbb{C}$ i.e., i and j are part of some exact sequences $0 \to N \xrightarrow{i} P \to P' \to 0$ and $0 \to N'' \to N \xrightarrow{j} M \to 0$, respectively.

Composition is also well defined (see [57]).

*1.1.5. Definition:* For n ≥ 0, define $K_n(\mathbb{C}) := \pi_{n+1}(BQ\mathbb{C})$ n ≥ 0, where for any topological space Y, $\pi_{n+1}(Y)$ is the (n+1)- homotopy group of Y.

*1.1.6 Note:* The definition above due to D.Quillen [57] coincides with the classical definition of $K_0(\mathbb{C})$ as the Abelian group on the isomorphism classes (C) of $\mathbb{C}$-objects subject to relations (C′) + (C″) = (C) wherever $0 \to C' \to C + C'' \to 0$ is an exact sequence in $\mathbb{C}$ (see [57]).

***1.2. The Plus Construction – Another Definition of $K_n(\mathcal{P}(A)) = K_n(A)$ n≥1***

There is an alternative definition of $K_n(\mathcal{P}(A)) = K_n(A)$, n ≥1 also due D. Quillen . This definition, which is also very useful for computations, arises from the following theorem.

*1.2.1. Theorem* [65] Let X be a connected CW-complex, N a perfect normal subgroup of $\pi_1(X)$. Then there exists a CW-complex $X^+$ (depending on N) and a map $i : X \to X^+$ such that $(i)$ $i_* : \pi_1(X) \to \pi_1(X^+)$ is the quotient map $\pi_1(X) \to \pi_1(X)/N$ $(ii)$ For any $\pi_1(N)/N$ -module L, there is an isomorphism $i_* : H_*(X, i^*L) \to H_*(X^+, L)$ where $i^*L$ is L considered as a $\pi_1(X)$-module. $(iii)$ The Space $X^+$ is universal in the sense that if Y is any CW-complex and $f : X \to Y$ is a map such that $f_* : \pi_1(X) \to \pi_1(Y)$ satisfies $f_*(N)$ =0 then there exists a unique map $f^+ : X^+ \to Y$ such that $f^+ i = f$

*1.2.2. Definition* Now Let A be a ring with identity and put X=BGL(A) in above theorem. Then $\pi_1 BGL(A) = GL(A)$ contains E(A) as perfect normal subgroup. Hence by the theorem above, there exists a space $BGL(A)^+$. Define $K_n(A) = \pi_n(BGL(A)^+)$ for all n ≥1.

***1.3. Examples of $K_n$ of Ordinary And Equivariant Exact Categories***

***1.3.1**. Let $\mathcal{C}$=$\mathcal{P}$(A)*, the category of finitely generated projective modules over a ring A with identity. We write $K_n(A)$ for $K_n(\mathcal{P}(A) = \pi_{n+1}(\mathcal{B}\mathcal{Q}\mathcal{P}(A))$.

***1.3.2.** If C=$\mathcal{M}$ (A)*, the category of finitely generated modules over a Noetherian ring A. We write $G_n(A)$ for $K_n(\mathcal{M}(A)) = \pi_{n+1}(\mathcal{B}\mathcal{Q}\mathcal{M}(A))$.

*Note.* In 1.3.1 and 1.3.2 above, we shall be interested the group ring A=RG where G is a finite group and R is the ring of integers in a number field or p-adic field F. We have indeed identified $\mathcal{P}$ (RG), $\mathcal{M}$ (RG) with some categories of G-representations in Chapter I. Since RG is an R-order in the semi-simple F-algebra FG, we shall also be interested in $K_n(A)$, $G_n(A)$ where A is an R-order in a semi-simple F-algebra $\Sigma$. Recall that A is a sub ring of $\Sigma$ such that R is contained in the centre of A, A is a finitely generated R-module and $F \otimes_R A = \Sigma$.

**1.3.3.** Also the inclusion functor $\mathcal{P}$ (RG)→$\mathcal{M}$(RG) induces Abelian group homomorphism $K_n(RG) \to G_n(RG)$ n ≥ 0 which generalizes to higher dimensions the Cartan map $K_0(RG) \to G_0(RG)$ (see [6] or [39])

***1.3.4.** When* $\mathcal{C} = \mathcal{P}_R(RG)$, the category of RG-lattices where R is a commutative ring with identity, we shall write $G_n(R,G)$ for $K_n(\mathcal{P}_R(RG))$. It is well know that when R is regular, then $G_n(R,G) \approx G_n(RG)$ (see [39] [28]).

When R is a field of characteristic zero, and G a finite group, then $G_0(F,G) \simeq G_0(FG)$ coincides with the Abelian group of generalized characters $\chi$ : G →F and this provides the initial connection between representation theory and K-theory of the group-algebra FG.

***1.3.5**. Let X be a scheme*, and $\mathcal{P}$(X) the category of locally free sheaves of $O_X$ -modules (or equivalently the category $\mathcal{VB}$(X) of (algebraic) vector bundles on X. We shall write $K_n(X)$ for $K_n(\mathcal{P}(X))$ or $K_n(\mathcal{VB}(X))$: Recall that if X=spec(R) for a commutative ring R with identity, we shall recover $K_n(R)$ as in 1.3.1

**1.3.6** Let X be a Noetherian scheme and $\mathcal{M}$(X) the category of coherent sheaves of $O_X$ -modules. We shall write $G_n(X)$ for $K_n(\mathcal{M}(X))$ with the observation that if X= spec(R), we recover $G_n(R)$ as in 1.3.2

**1.3.7** Let G be an algebraic group over a field F (i.e. an F-group), X a G-scheme, $\mathcal{M}$(G,X) the category of coherent G-modules that are also coherent as $O_X$ -modules. We write $G_n(G,X)$ for $K_n(\mathcal{M}(G,X)$ for all n ≥ 0. If A is a finite dimensional separable F-algebra, X a G-scheme, $\mathcal{M}$(G,X,A) as defined in I,3.3.3 (b), we shall write $G_n(G,X,A)$ for $K_n(\mathcal{M}(G,X,A)$ for all n ≥ 0

**1.3.8** If G is an F-group and X a G-scheme and $\mathcal{P}$ (G,X) as in I 3.3.2, then we shall write $K_n(G,X)$ for $K_n(\mathcal{P}(G,X))$

*Note* that $\mathcal{P}(G,X)$ can be identified with the category $\mathcal{V}B_G(X)$ of G-equivariant (algebraic) vector bundles on X and so $K_n(G,X) \simeq K_n(\mathcal{P}(G,X)) \simeq K_n\mathcal{V}B_G(X)$ for all n ≥ 0. We shall also write this group as $K_n^G(X)$

If A is a finite dimensional separable F-algebra, and $\mathcal{P}(G,X,A)$ as in I3.3.3 (b), we shall write $K_n(G,X,A)$ for $K_n(\mathcal{P}(G,X,A))$ n ≥0

**1.3.9** Let G be an F - group and H a closed subgroup of G. In I,3.2.2 we saw that there is an equivalence of categories between the exact category $\mathcal{P}(F)_H$ or equivalently $\mathrm{Rep}_F(H)$ of representation of H in $\mathcal{P}$(F) and the exact category $\mathcal{V}B_G(G/H)$ of G-equivariant vector bundles on G/H. Note that $K_0(\mathcal{P}(F)_H) \simeq K_0(\mathcal{VB}_G(G/H))$ where the latter group is denoted by $K_0^G(G/H)$. It is also usual to denote the “representation group” of H (i.e. the group of generalized characters of H) by R(H). In fact R (H) is a ring called representation ring of H. (see [64]) .Note that for any algebraic group G over F, $K_0(\mathrm{Rep}_F(G) \simeq R(G)$ is a free Abelian group generated by the classes of irreducible representations and that R (G) also has the structure of a ring induced by tensor product.

**1.3.10** Now, in the notation of I 3.2.3, let $\tilde{G}$ a semi-simple connected and simply connected, F-split algebraic group over F, $\tilde{T} \subset \tilde{G}$ be a maximal F-split torus of $\tilde{G}$, $\tilde{P} \subset \tilde{G}$ a parabolic subgroup of $\tilde{G}$ containing the torus $\tilde{T}$, $\mathcal{F} = \tilde{G}/\tilde{P}$ the flag variety. Let $W = N_{\tilde{G}}\left(\tilde{T}\right)/\tilde{T}$ be the Weyl group of $\tilde{G}$ (a finite group), $W_{\tilde{P}} : \{w \in W| \, w\tilde{P}w^{-1} = \tilde{P}\}$ put $s(\mathcal{F}) = [W : W_{\tilde{P}}]$. Then we have the following.

***1.3.11***. *Theorem* [55] $R(\tilde{P})$ is a free $R(\tilde{G})$-module of rank $s(\mathcal{F})$.

**1.3.12** In the notation of I,3.2.3 let $\mathcal{VB}_{\tilde{G}}(\mathcal{F}:\mathcal{X})$ be the full subcategory of $\mathcal{VB}_{\tilde{G}}(\mathcal{F})$ consisting of those $\tilde{V}$ such that $\tilde{Z}$ acts on every fibre of $\tilde{V}$ by the character $\mathcal{X}$. We shall write $K_n^G(\mathcal{F},\mathcal{X})$ for $K_n(\mathcal{VB}_{\tilde{G}}(\mathcal{F},\mathcal{X}))$ and $R^{\mathcal{X}}(\tilde{P})$ for $K_0(\mathrm{Re}\, p_F^{\mathcal{X}}(\tilde{P}))$

**1.3.13** Let $\tilde{G}$, $\tilde{Z}$, $\tilde{T}$, $\tilde{P}$ be as in I,3.2.3 put $G=\tilde{G}/\tilde{Z}$, $P=\tilde{P}/\tilde{Z}$, $T=\tilde{T}/\tilde{Z}$, $\mathcal{F}=\tilde{G}/\tilde{P}=G/P$ put $\mathcal{G}=Gal(F_{sep}/F)$ where $F_{sep}$ is the separable closure of F. Let c: $\mathcal{G}\to G(F_{sep})$ be the 1 – co cycle (see [55]) and ${}_c\mathcal{F}$ the twisted form of $\mathcal{F}$ corresponding to c (see [55]. [40])

We shall write $K_n^G({}_c\mathcal{F})$ for $K_n(\mathcal{VB}_G({}_c\mathcal{F}))$

**1.3.14.** Let B be a finite dimensional separable F-algebra, X a smooth projective variety equipped with the action of an affine algebraic group G over F, ${}_cX$ the twisted form of X via a 1-cocycle c. Let $\mathcal{VB}_G({}_cX,B)$ be the category of vector bundles on ${}_cX$ equipped with left B-module structure. We write $K_n^G({}_cX,B)$ for $K_n(\mathcal{VB}_G({}_cX,B))$.

**1.3.15.** a) Let G be a finite group, S a G-set and $\underline{S}$ the translation category of S (see I,2.3.2), $\mathcal{C}$ an exact category. We saw in I, 2.3.3 that the category $[\underline{S},\mathcal{C}]$ of covariant functor $\zeta:\underline{S}\to\mathcal{C}$ is an exact category. For all n ≥ 0, let $K_n^G(S,\mathcal{C})$ be the $n^{th}$ algebraic K-group of the category $[\underline{S},\mathcal{C}]$ with respect to fibre wise exact sequences.

Recall that if $\mathcal{C}=\mathcal{M}(R)$ and S=G/H, H ≤ G, Then $[G/H,\mathcal{M}(R)]\simeq\mathcal{M}(RH)$ and so $K_n^G(G/H,\mathcal{M}(R))\simeq K_n(\mathcal{M}(RH))\simeq G_n(RH)$ for all n ≥ 0

If $\mathcal{C}=\mathcal{P}(R)$ then $[G/H,\mathcal{P}(R)]\simeq\mathcal{P}_R(RH)$ and so $K_n^G(G/H,\mathcal{P}(R))\simeq K_n(\mathcal{P}_R(RH))\simeq G_n(R,H)$ for all n ≥ 0

**1.3.16.** Let S, T be G-sets, $\mathcal{C}$ an exact category. Recall from I 2.4.1, that we obtained an exact category ${}^T[\underline{S},\mathcal{C}]$ as follows: $ob({}^T[\underline{S},\mathcal{C}])=ob[\underline{S},\mathcal{C}]$ while exact sequences in ${}^T[\underline{S},\mathcal{C}]$ are T-exact sequences in $[\underline{S},\mathcal{C}]$. We now denote by $K_n^G(S,\mathcal{C},T)$ the $n^{th}$ algebraic K-group $K_n({}^T[\underline{S},\mathcal{C}])$.

*Note* that $K_n^G(G/H,\mathcal{P}(R),T)$ (resp $K_n^G(G/H,\mathcal{M}(R),T)$ is the $n^{th}$ algebraic K-group of $\mathcal{P}_R(RH)$ $resp(\mathcal{M}(RH))$ with respect to exact sequences that split when restricted to the various subgroups H´ of H such that $T^{H'}\neq 0$

**1.3.17.** Let S, T be G-sets, $\mathcal{C}$ an exact category. Recall from I, 2.4.3 that we have an exact category $[\underline{S},\mathcal{C}]_T$ of T-projective functors in $[\underline{S},\mathcal{C}]$ with respect to split exact sequences. We write $P_n^G(S,\mathcal{C},T)$ for $K_n([\underline{S},\mathcal{C}]_T)$

*Note* that if T=G/e where e is the identity element of G, then $\left[G/H,\mathcal{P}(R)\right]_{G/e}\simeq\mathcal{P}(RH)$ and $K_n(\left[G/H,\mathcal{P}(R)\right]_{G/e})\simeq K_n(RH)$ for all n ≥ 0.

### *1.4. Mod-$l^s$ higher K-theory (ordinary and equivariant)*

**1.4.1**. Let $\mathcal{C}$ be an exact category, $\ell$ a rational prime, s a positive integer, $M^{n+1}_{\ell^s}$ the (n+1) - dimensional mod-$l^s$ space i.e. the space obtained from $S^n$ by attaching an (n+1)-cell via a map of degree $\ell^s$ (see [5], [53])

If X is an H-space, we write $\pi_{n+1}(X,\mathbb{Z}/\ell^s)$ for $\left[M^{n+1}_{i^s},X\right]$, the set of homotopy classes of maps from $M^{n+1}_{\ell^s}$ to X. If $\mathcal{C}$ is an exact category and X=BQ$\mathcal{C}$, we write $K_n(\mathcal{C},\mathbb{Z}/\ell^s)$ for $\pi_{n+1}(BQ\mathcal{C},\mathbb{Z}/\ell^s)$ for n ≥ 1 and $K_0(\mathcal{C},\mathbb{Z}/\ell^s)$ for $K_0(\mathcal{C})\otimes\mathbb{Z}/\ell^s$ . Call $K_n(\mathcal{C},\mathbb{Z}/\ell^s)$ mod-$l^s$ K-theory of C.

### 1.4.2. Examples

(i) If A is a ring with identity, and $\mathcal{C}=\mathcal{P}$ (A) the category of finitely generated projective A-modules, write $K_n(A,\mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(A),\mathbb{Z}/l^s)$ . Note that $K_n(A,\mathbb{Z}/l^s)$ is also $\pi_n(BGL(A)^+,\mathbb{Z}/l^s)$.

(ii) If Y is a scheme and $\mathcal{C}=\mathcal{P}$ (Y), the category of locally free sheaves of $O_Y$ -modules, write $K_n(Y,\mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(Y),\mathbb{Z}/l^s)$. Note that for Y=Spec(A), A commutative, we recover $K_n(A,\mathbb{Z}/l^s)$.

(iii) Let A be a Noetherian ring and $\mathcal{M}$ (A) the category of finitely generated A-modules. We write $G_n(A,\mathbb{Z}/l^s)$ for $K_n(\mathcal{M}(A),\mathbb{Z}/l^s)$.

(iv) If Y is a Noetherian scheme, $\mathcal{C}=\mathcal{M}$(Y) the category of coherent sheaves of $O_Y$ -modules, write $G_n(Y,\mathbb{Z}/l^s)$ for $K_n(\mathcal{M}(Y),\mathbb{Z}/l^s)$.

(v) Let G be algebraic group over field F, X a G-scheme and $\mathcal{C}=\mathcal{M}$ (G,X). Write $G_n((G,X),\mathbb{Z}/l^s)$ for $K_n(\mathcal{M}(G,X),\mathbb{Z}/l^s)$

(vi) If $\mathcal{C}=\mathcal{P}$(G,X) write $K_n((G,X),\mathbb{Z}/l^s)$ for $K_n(\mathcal{P}(G,X),\mathbb{Z}/l^s)$

(vii) If $\mathcal{C}=\mathcal{VB}_G({}_cX,B)$, we write $K_n(({}_cX,B),\mathbb{Z}/l^s)$ for $K_n(\mathcal{VB}_G({}_cX,B),\mathbb{Z}/l^s)$

(viii) If $\mathcal{C}=\mathcal{M}$(G,X,A) we Write $G_n((G,X,A),\mathbb{Z}/l^s)$ for $K_n(\mathcal{M}(G,X,A),\mathbb{Z}/l^s)$

### *1.5. Profinite Higher K-Theory (Ordinary and Equivariant)*

1.5.1. Now put $M_{\infty}^{n+1} = \underset{\longrightarrow s}{\lim} M_{\ell^s}^{n+1}$, We define the profinite K-theory of an exact category $\mathbb{C}$ by $K_n^{pr}\left(\mathbb{C}, \hat{\mathbb{Z}}_\ell\right) = \left[M_{l^\infty}^{n+1}; BQ\mathbb{C}\right]$. We also write $K_n\left(\mathbb{C}, \hat{\mathbb{Z}}_\ell\right)$ for $\underset{\longleftarrow s}{\lim} K_n(\mathbb{C}, \mathbb{Z}/l^s)$

**1.5.2.Examples**

(i) If $\mathbb{C} = \mathcal{P}(A)$, we write $K_n^{pr}\left(A, \hat{\mathbb{Z}}_\ell\right)$ for $K_n^{pr}\left(\mathcal{P}(A), \hat{\mathbb{Z}}_\ell\right)$ and $K_n\left(A, \hat{\mathbb{Z}}_\ell\right) = K_n\left(\mathcal{P}(A), \hat{\mathbb{Z}}_\ell\right)$ for A any ring

(ii) If $\mathbb{C} = \mathcal{P}(Y)$, we write $K_n^{pr}\left(Y, \hat{\mathbb{Z}}^l\right)$ for $K_n^{pr}\left(\mathcal{P}(Y), \hat{\mathbb{Z}}_l\right)$ and $K_n\left(Y, \hat{\mathbb{Z}}_l\right)$ for $K_n\left(\mathcal{P}(Y), \hat{\mathbb{Z}}_l\right)$,Y a scheme

(iii) If $\mathbb{C} = \mathcal{M}(A)$, we write $G_n^{pr}(A, \hat{\mathbb{Z}}_l)$ for $G_n^{pr}\left(\mathcal{M}(A), \hat{\mathbb{Z}}_l\right)$ and $G_n\left(A, \hat{\mathbb{Z}}_\ell\right)$ for $K_n\left(\mathcal{M}(A), \hat{\mathbb{Z}}_l\right)$ , A a Noetherian ring

(iv) If $\mathbb{C} = \mathcal{M}(Y)$, write $G_n^{pr}\left(Y, \hat{\mathbb{Z}}_l\right)$ for $K_n^{pr}\left(\mathcal{M}(Y), \hat{\mathbb{Z}}_l\right)$ $G_n(Y, \hat{\mathbb{Z}}_\ell)$ for $K_n(\mathcal{M}(Y), \hat{\mathbb{Z}}_\ell)$ Y a Noetherian Scheme

(v) If $\mathbb{C} = \mathcal{M}(G,X)$, write $G_n^{pr}\left((G, X), \hat{\mathbb{Z}}_l\right)$ for $K_n^{pr}\left(\mathcal{M}(G, X), \hat{\mathbb{Z}}_l\right); G_n((G, X), \hat{\mathbb{Z}}_\ell)$ for $K_n(\mathcal{M}(G, X), \hat{\mathbb{Z}}_\ell)$ G an algebraic group, Z a G-Scheme.

(vi) If $\mathbb{C} = \mathcal{P}(G,X)$, write $K_n^{pr}\left((G, X), \hat{\mathbb{Z}}_l\right)$ for $K_n^{pr}\left(\mathcal{P}(G, X), \hat{\mathbb{Z}}_l\right)$ $G_n((G, X), \hat{\mathbb{Z}}_l)$ for $K_n(\mathcal{M}(G, X), \hat{\mathbb{Z}}_\ell)$ X a G – scheme, G an algebraic group

(vii) If $\mathbb{C} = \mathcal{VB}_G({}_cX, B)$, write $K_n^{pr}\left(({}_cX, B), \hat{\mathbb{Z}}_l\right)$ for $K_n^{pr}\left(\mathcal{VB}_G({}_cX, B), \hat{\mathbb{Z}}_l\right); K_n({}_cX, B), \hat{\mathbb{Z}}_\ell)$ for $K_n(\mathcal{VB}_G({}_cX, B), \hat{\mathbb{Z}}_\ell)$ (See earlier definitions)

(viii) If $\mathbb{C} = \mathcal{M}$ (G, X, A), write $G_n^{pr}\left((G, X, A), \hat{\mathbb{Z}}_l\right)$ for $K_n^{pr}\left(\mathcal{M}(G, X, A), \hat{\mathbb{Z}}_l\right)$ .and $G_n((G, X, A), \hat{\mathbb{Z}}_\ell)$ for $K_n(\mathcal{M}(G, X, A), \hat{\mathbb{Z}}_\ell)$ ( see earlier definitions)

(ix) If $\mathcal{C} = \mathcal{P}(G,X,A)$, write

$K_n^{pr}\left((G,X,A),\hat{\mathbb{Z}}_l\right)$ for $K_n^{pr}\left(\mathcal{P}(G,X,A),\hat{\mathbb{Z}}_l\right)$ $K_n((G,X,A),\hat{\mathbb{Z}}_\ell)$ for

$K_n(\mathcal{P}(G,X,A)\widehat{\mathbb{Z}}_l)$

## Section 2. Induction Techniques for finite group actions ; Mackey functors

### *2.1. Mackey functors – Brief Review*

In this section, we briefly introduce Mackey functors in a way relevant to our context. For more general definition and presentation see ([39] [27])

Mackey functors are functors satisfying certain functorial properties in particular, the categorical version of Mackey subgroup theorem in representation theory. Induction theory has always aimed at computing various invariants of certain classes of subgroups of a group G. It turns out that for such a Mackey functor M, one can always find a canonical smallest class $\mathcal{U}_M$ of subgroup of G such that the values of M an any G-set can be computed from their restrictions to the full subcategory of G-sets of the form G/H, H$\in \mathcal{U}_M$ . For more details (see [39] [27])

#### 2.1.1. Definition

Let G be a finite group, G$\mathcal{S}$et the category of (finite) G-sets. A pair $(M_*, M^*)$ of functors G$\mathcal{S}$et $\to$R-$\mathcal{M}$od is a Mackey functor if

(i) $M_*$: G$\mathcal{S}$et $\to$R-$\mathcal{M}$od is covariant and $M^*$: GSet $\to$ R – $\mathcal{M}$od is contravariant and $M_*(X) = M^*(X) = M(X)$ for any G-set X

(ii) $M^*$ transforms finite disjoint unions in GSet into finite products in R-$\mathcal{M}$od, i.e., the embeddings $X_i \to \dot{\cup} X_i$ induce isomorphism $M(X_1 \dot{\cup} X_2 \dot{\cup} \ldots \dot{\cup} X_n) \simeq M(X_1)\times M(X_2)\times \ldots \times M(X_n)$

(iii) For any pull-back diagram

$$\begin{array}{ccc} X_1 \underset{Y}{\times} X_2 & \xrightarrow{p_2} & X_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow{f_1} & Y \end{array}$$

in G/Set, The diagram

$$\begin{array}{ccc} M(X_1 \underset{Y}{\times} X_2) & \xrightarrow{M_*(p_2)} & M(X_2) \\ \uparrow{\scriptstyle M^*(p_1)} & & \uparrow{\scriptstyle M^*(f_1)} \\ M(X_1) & \xrightarrow{M_*(f_1)} & M(Y) \end{array}$$

commutes (Mackey subgroup property).

*2.1.2. A morphism (or natural transformation) of Mackey functors* $\tau : M \to N$ consists of a family of homomorphism $\tau(X): M(X) \to N(X)$, indexed by the objects X in G/Set, such that $\tau$ is a natural transformation of $M_*$ as well as of $M^*$, i.e. such that for any G-map $f : X \to Y$ the diagrams.

$$\begin{array}{ccc} M(X) & \xrightarrow{M_*(f)} & M(Y) \\ \downarrow{\scriptstyle \tau(X)} & & \downarrow{\scriptstyle \tau(Y)} \\ N(X) & \xrightarrow{N_*(f)} & N(Y) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M(Y) & \xrightarrow{M^*(f)} & M(X) \\ \downarrow{\scriptstyle \tau(Y)} & & \downarrow{\scriptstyle \tau(X)} \\ N(Y) & \xrightarrow{N^*(f)} & N(X) \end{array}$$

are commutative.

*2.1.3. A pairing $M \times N \to L$* of two Mackey functors M and N into a third one, called L, is a family of R-bilinear maps

$$M(X) \times N(X) \to L(X) : (m, n) \to m \square n$$

such that for any G map $f : X \to Y$ the following diagrams commute

$$\begin{array}{ccc} M(Y) \times N(Y) & \longrightarrow & L(Y) \\ \downarrow{\scriptstyle M^*(f) \times N^*(f)} & & \downarrow{\scriptstyle L^*(f)} \\ M(X) \times N(X) & \longrightarrow & L(X) \end{array}$$

$$\begin{array}{ccccc} M(X)\times N(Y) & \xrightarrow{Id\times N^*(f)} & M(X)\times N(X) & \longrightarrow & L(X) \\ \downarrow{\scriptstyle M_*(f)\times Id} & & & & \downarrow{\scriptstyle L^*(f)} \\ M(Y)\times N(Y) & & \longrightarrow & & L(Y) \end{array}$$

$$\begin{array}{ccccc} M(Y)\times N(X) & \xrightarrow{M^*(f)\times Id} & M(X)\times N(X) & \longrightarrow & L(X) \\ \downarrow{\scriptstyle Id\times N_*(f)} & & & & \downarrow{\scriptstyle L_*(f)} \\ M(Y)\times N(Y) & & \longrightarrow & & L(Y) \end{array}$$

(the last two being related to Frobenius reciprocity)

2.1.4. A Green functor is a Mackey functor $\mathcal{G}$: G$\mathcal{S}$et →R → $\mathcal{M}$od together with pairing $\mathcal{G}$ x $\mathcal{G}$ → $\mathcal{G}$ such that an R-bilinear map $\mathcal{G}$(X) x $\mathcal{G}$(X) → $\mathcal{G}$(X) turns $\mathcal{G}$(X) into an R-algebra with unit $1_{\mathcal{G}(x)}$ and such that for each G-map $f : X \to Y$ , the equation

$f^*(Y)(1_{\mathcal{G}(Y)}) = 1_{\mathcal{G}(X)}$ holds.

2.1.5. If $\mathcal{G}$ is a green functor, M a Mackey functor and $\mathcal{G}$ x M → M a pairing such that $1_{\mathcal{G}(X)}$ acts as identify on $\mathcal{M}$(X),we shall call M with respect to this pairing a $\mathcal{G}$-module.

### *2.2. Higher K-Theory as Mackey Functors (For Finite Group Actions)*

*2.2.1. Let G be finite group, S, T, G-sets; and $\mathcal{C}$ an exact category*. In I, section 2 and section 1 of this chapter, we obtained three equivariant exact categories with associated higher K-groups as follows:

(1) $K_n^G(S,\mathcal{C})$ is the $n^{th}$ algebraic K-group (n ≥0) of the exact category $\left[\underline{S},\mathcal{C}\right]$ with respect to fibre-wise exact sequences. Recall that if $\mathcal{C}$=$\mathcal{P}$(R), S = G/H, then $K_n^G(G/H,\mathcal{P}(R)) \simeq G_n(R,H)$ and that if C = $\mathcal{M}$(R), then $K_n^G(G/H,\mathcal{M}(R)) \simeq G_n(RH)$ and that when R is regular $G_n(R,H) \simeq G_n(RH)$

(2) $K_n^G(S,\mathcal{C},T)$ is the $n^{th}$ algebraic K-group (n ≥ 0) of the exact category ${}^T\left[\underline{S},\mathcal{C}\right]$ with respect to T-exact sequences in $\left[\underline{S},\mathcal{C}\right]$. Note that when $\underline{S}$ = G/H, $\mathcal{C}$=$\mathcal{M}$(R), (resp. $\mathcal{C}$= $\mathcal{P}$(R)) then $K_n^G(G/H,\mathcal{M}(R),T)$ *resp.*$K_n^G(G/H,\mathcal{P}(R),T))$ is the $n^{th}$ algebraic K-group of the exact category $\mathcal{M}$(RH) (resp $\mathcal{P}_R(RH)$ with respect to exact sequences which split when restricted to the various subgroups, H´ of H such that $T^{H'} \neq \phi$ (recall that $T^{H'} = \{t \in T / gt = t$ for all g∈ H´}

(3) $P_n^G(S,\mathbb{C},T)$ is the $n^{th}$ algebraic K-group (n ≥ 0) of the exact category $[\underline{S},\mathbb{C}]_T$ with respect to split exact sequences.

(4) If S = G/H, $\mathbb{C}$=$\mathcal{P}$(R) (resp $\mathcal{M}$(R)), then $P_n^G(G/H,\mathcal{P}(R),T)$ (resp $P_n^G(G/H,\mathcal{M}(R),T)$) is the $n^{th}$ algebraic K-group of the exact category $\mathcal{P}_R(RH)$ (resp $\mathcal{M}(RH)$) consisting of objects that are relatively H´ - projective for subgroups H´ of H such that $T^{H'} \neq \phi$ with respect to split exact sequences.

Note in particular that $P_n^G(G/H,\mathcal{P}(R),G/e) = K_n(RH)$. For details and properties of this construction see [39]. [10] We now have the following.

**2.2.2. Theorem [10] [39]**

Let G be a finite group, T a G-set, $\mathbb{C}$ an exact category, $\mathcal{A}$b the category of Abelian groups .Then $K_n^G(-,\mathbb{C})$, $K_n^G(-,\mathbb{C},T)$, $P_n^G(-,\mathbb{C},T)$: GSet → Ab are Mackey functors for all n ≥ 0. If the pairing $\mathbb{C}$ x $\mathbb{C}$ → $\mathbb{C}$ is naturally associative and commutative and contains a natural unit, then $K_0^G(-,\mathbb{C})$, $K_0^G(-,\mathbb{C},T)$ G$\mathcal{S}$et → $\mathcal{A}$b are Green functors ; $K_n^G(-,\mathbb{C})$ is a unitary $K_0^G(-,\mathbb{C})$-module and $K_n^G(-,\mathbb{C},T)$ and $P_n^G(S-,\mathbb{C},T)$ are $K_0^G(-,\mathbb{C},T)$-modules.

For a proof see [10].[39].

**2.2.3. Remarks**

(1) It is well known that the Burnside functor Ω: G$\mathcal{S}$et → $\mathcal{A}$b is a Green functor and that any Mackey functor M : G$\mathcal{S}$et → $\mathcal{A}$b is an Ω-module and any Green Functor is an Ω-algebra (see [39],[7], [27]). Hence the above K-functors $K_n^G(-,\mathbb{C},T)$, $P_n^G(S-,\mathbb{C},T)$ and $K_n^G(-,\mathbb{C})$ are Ω-modules, and $K_0^G(-,\mathbb{C},T)$ and $K_0^G(-,\mathbb{C})$ are Ω-algebra.

(2) Let M be any Mackey functor : G$\mathcal{S}$et →$\mathcal{A}$b, X a G-set. Define $K_M(X)$ as the kernel of M(G/G) → M(X) and $I_M(X)$ as the image of M(X) → M(G/G). An important induction result is that |G| $\mathcal{M}$ (G/G) $\subseteq K_M(X) + I_M(X)$ for any Mackey functor M and G-set X. ([39] [7]). This result also applies to all the K-theoretic functors defined above.

(3) If M is any Mackey functor: G$\mathcal{S}$et →$\mathcal{A}$b; X a G-set, define a Mackey functor $M_X$ :G$\mathcal{S}$et → $\mathcal{A}$b by $M_X(Y) = M(X \times Y)$. The projection map pr: $X \times Y \to Y$ defines a natural transformation $\theta_X : M_X \to M$ where $\theta_X(Y)$ = pr: $M(X \times Y) \to M(Y)$. M is said to be X-projective if $\theta_X$ is split surjective (see

[39], [27]). Now define the defect base $D_M$ of M by $D_M = \{H \leq G | X^H \neq \phi\}$ where X is a G-set (called the defect set of M) such that M is Y-projective iff there exists a G-map $f : X \to Y$ (see [39]). If M is a module over Green functor $\mathcal{G}$, then M is X-projective iff G is X-projective iff the induction map $\mathcal{G}$(X) →$\mathcal{G}$(G/G) is surjective. In general, proving induction results reduce to determining G-sets X for which $\mathcal{G}$(X) →$\mathcal{G}$(G/G) is surjective and this in turn reduces to computing $D_{\mathcal{G}}$. Thus, one could apply induction techniques to obtain results on higher K-groups which are modules over the Green functor $K_0^G(-,\mathbb{C})$ and $K_0^G(-,\mathbb{C},Y)$ for suitable exact categories $\mathbb{C}$ e.g. $\mathbb{C} = \mathcal{P}$(R) or $\mathcal{M}$(R) (see [39])

(4) One can show via general induction theory principles that for suitably chosen $\mathbb{C}$, all the higher K-functor $K_n^G(-,\mathbb{C})$, $K_n^G(-,\mathbb{C},T)$ are ''hyper elementary computable'' – see ([39], [24]) or below.

### *2.3. Some Consequent Results on Higher K-Theory of Grouprings*

In this subsection, we provide some specific results on higher K-theory of grouprings that are consequences of the techniques briefly outlined in 2.1 and 2.2. For proofs of these results, see [39].

**2.3.1. Theorem [39] [10]**

Let G be a finite group, $T$ $a$ G-set, k a field of characteristic $p$, $p \neq 0$ then, there exists an isomorphism of Mackey functors :

$$\mathbb{Z}\left(\frac{1}{p}\right)\otimes P_n^G\left(-,\mathcal{P}(k),T\right) \simeq \mathbb{Z}\left(\frac{1}{p}\right)\otimes K_n^G\left(-,\mathcal{M}(k),T\right): G\mathcal{S}ets \to \mathcal{A}b.$$ Hence for all n ≥ 0, the Cartan map $K_n(kG) \to G_n(kG)$ induces isomorphismes

$$\mathbb{Z}\left(\frac{1}{p}\right)\otimes K_n(kG) \to \mathbb{Z}\left(\frac{1}{p}\right)\otimes G_n(kG)$$

Proof : see [39] [9]

We also have the following consequences of 2.3.1.

**2.3.2. Theorem [39] [9]**

Let p be a rational prime, k a field of characteristic p, G a finite group. Then for all n ≥ 1 $K_{2n}(kG)$ is a finite p-group

The Cartan homomorphism $\varphi_{2n-1} : K_{2n-1}(kG) \to G_{2n-1}(kG)$ is surjective and Ker $\varphi_{2n-1}$ is the Sylow p-subgroup of $K_{2n-1}(kG)$

2.3.3. To be able to state the next results, we need an alternative characterization of Mackey functors as functors on a category of subgroups of G rather than on G-sets.

Let $\underline{\delta G}$ denote the subgroup category whose objects are the various subgroups of G with $\underline{\delta G}(H_1,H_2)=\{(g,H_1,H_2), g\in G, gH_1g^{-1}\le H_2\}$ and composition of $(g,H_1,H_2)\in\delta G(H_1,H_2)$ and $(h,H_2,H_3)\in\underline{\delta G}(H_2,H_3)$ defined by $(h,H_2,H_3)\circ(g,H_1,H_2)=(hg,H_1,H_3)$, so that $(e,H,H)\in\delta G(H,H)$ is the identity where H ≤ G and e ∈ G is the trivial element.

There is a canonical functor C, the coset functor from $\underline{\delta G}$ into G$\mathcal{S}$et : H →G/H and with each morphism $(g,H_1,H_2)\in\delta G(H_1,H_2)$, the G-map $\Psi_{g-1}:G/H_1\to G/H_2 : xH_1\to xg^{-1}H_2$.

If M: G$\mathcal{S}$et → $\mathcal{A}$b is a Mackey functor , then the composite $\hat{M}=M\circ C:\underline{\delta G}\to\mathcal{A}b$ is a bifunctor which can be shown to describe situations similar to the Mackey subgroup theorem (see [11] for details). Call $\hat{M}$ a G-functor as so christened by J.A Green (see [11])

It can be shown that there is a one-one correspondence between the G-functors $\hat{M}:\underline{\delta G}\to\mathcal{A}b$ and Mackey functors M: G$\mathcal{S}$ets →$\mathcal{A}$b. So we can identify any Mackey functor M: G$\mathcal{S}$et → $\mathcal{A}$b with $\hat{M}:\underline{\delta G}\to\mathcal{A}b$ and thus sometimes write M(H) instead of M(G/H). (See [39]). To be able to state the next result we need the following definition.

**Definition 2.3.4.**

Let G be a finite group, $\mathcal{U}$ a collection of subgroups of G closed under subgroups and isomorphic images, A a commutative ring with identity. Then a Mackey functor. $M:\underline{\delta G}\to$ A-$\mathcal{M}$od is said to be $\mathcal{U}$-computable if the restriction maps $M(G)\to\prod_{H\in\mathcal{U}}M(H)$ induces an isomorphism $M(G)\to\varprojlim_{H\in\mathcal{U}}M(H)$ where $\varprojlim_{H\in\mathcal{U}}M(H)$ is the subgroup of all $(x)\in\prod_{H\in\mathcal{U}}M(H)$ such that for any H, H´∈$\mathcal{U}$, and g ∈G with $gH'g^{-1}\le H$, $\varphi:H'\to H$ given by $h\to ghg^{-1}$, then $M(\varphi)(x_H)=x_{H'}$.

Now, if A is a commutative ring with identity, $M:\underline{\delta G}\to\mathcal{A}b$ a Mackey functor, then $A\otimes\mathcal{M}:\underline{\delta G}\to A$-$\mathcal{M}$od is also a Mackey functor where $(A\otimes M)(H)=A\otimes M(H)$.

Now, let $\mathcal{P}$ be a set of rational primes, $\mathbb{Z}_{\mathcal{P}}=\mathbb{Z}\left[\frac{1}{q} \mid q\notin\mathcal{P}\right]$ ,$\mathcal{C}$(G) the collection of all cyclic subgroups of G, $h_{\mathcal{P}}\mathcal{C}(G)$ the collection of all $\mathcal{P}$-hyperelementary subgroups of G, ie $h_{\mathcal{P}}\mathcal{C}(G)=\{H\le G \mid$ there exists $H'\le H, H'\in\mathcal{C}(G), H/H'$ is a p-group for some p ∈$\mathcal{P}$}

Recall that if R is Dedekind domain with quotient field F, G a finite group, we define for all n ≥ 0

$SK_n(RG) := Ker(K_n(RG) \to K_n(FG)$, $SG_n(RG) := Ker(G_n(RG) \to G_n((FG))$

We now have the following result.

**2.3.5. Theorem [39] [24]**

Let R be aDedekind ring , G a finite group, M any of the modules $K_n(R-)$, $G_n(R-)$, $SG_n(R-)$ over $G_0(R-)$ then $\mathbb{Z}_{\mathfrak{P}} \otimes M$ is $h_{\mathfrak{P}}(\mathbb{C}(G))$-computable

# CHAPTER III. SOME RESULTS ON THE ACTION OF FINITE AND ALGBRAIC GROUPS

## Section 1. Some results on $K_n(RG), G_n(RG), C\ell_n(RG), SK_n(RG), SG_n(RG)$ n ≥ 0 (G finite) and consequences for some infinite groups

### *1.1. On $K_n(RG)$, $G_n(RG)$, $SK_n(RG)$, $SG_n(RG)$ , G finite*

1.1.1. Let R be a Dedekind ring with quotient field F, G a finite group. Recall that $K_n(RG) := K_n(\mathcal{P}(RG))$ and $G_n(RG) := K_n(\mathcal{M}(RG))$ where we have earlier identified $\mathcal{P}(RG)$ and $\mathcal{M}(RG)$ as categories of G-representations. Hence , the study of $K_n(RG)$, $G_n(RG)$ belongs to integral representation theory

*Define:* $SK_n(RG) := Ker(K_n(RG) \to K_n(FG)$

$SG_n(RG) := Ker(G_n(RG) \to G_n((FG))$ for all n ≥ 0

*Define:* $C\ell_n(RG) = Ker(SK_n(RG)) \to \underset{\underline{p}}{\oplus} SK_n(\hat{R}_{\underline{p}}G))$ where R is the ring of integers in a number field F and where $\underline{p}$ runs through all prime ideals of R and $\hat{R}_{\underline{p}}$ is the completion of R at $\underline{p}$ . Call $C\ell_n(RG)$ the n-dimensional (higher) class group of RG. Note that $C\ell_o(RG)$ coincides with the usual class goups $C\ell(RG)$ of RG. (See [39], [6])

We shall provide in this section some important results on $K_n(RG)$, $G_n(RG)$, $SK_n(RG)$, $SG_n(RG)$ and $C\ell_n(RG)$ which constitute the core results of studies on higher K-theory of integral grouprings.

**1.1.2. Remarks**

We shall be interested mostly in R being the ring of integers in a number field or a p-adic field. Note that the phenomenal growth of K-theory has been due partly to the fact that the classical K-groups of groupings (i.e. above groups for n = 0; 1,2) house various interesting topological/geometric invariants . For example

(1) $C\ell_0(RG) = C\ell(RG)$ house Swan-Wall invariants see [74].

(2) $K_1(RG), SK_1(RG)$ house Whitehead torsion and is also useful in the classification of manifolds (see [54] [51])

(3) $K_2(RG)$ helps in the understanding of pseudo-isotopy of manifolds. See [15], [39]

First we give finiteness results on $K_n(RG)$, $G_n(RG)$, $SK_n(RG)$, $SG_n(RG)$, $C\ell_n(RG)$. Note that these results are proved in [39] in the generality of arbitray R-orders $\Lambda$ in a semi-simple F-algebra $\Sigma$, which invariably apply to $\Lambda = RG$

**1.1.3. Theorem [39] [28] [37]**

Let R be the ring of integers in a number field F, G a finite group. Then for all n ≥ 1

$K_n(RG)$ is finitely generated Abelian group and $K_{2n}(RG)$ is finite.

$SK_n(RG)$ is a finite group. Hence $C\ell_n(RG)$ is finite.

$SK_n(\hat{R}_{\underline{p}}G)$ is also finite.

If G is a finite p-group, then $SK_n(RG), SK_n(\hat{R}_{\underline{p}}G)$ are finite p-groups

(See [39] [28] [37] ) for the proofs.)

**1.1.4. Theorem [39] [47] [24]**

Let R be the ring of intergers in a number field F, G a finite group. Then for all n ≥ 1

$G_n(RG)$ is finitely generated Abelian group

$SG_n(RG) = 0$

$SG_n(\hat{R}_{\underline{p}}G) = 0$ where $\underline{p}$ is prime ideal of $R$ and $\hat{R}_{\underline{p}}$ is the completion of R at $\underline{p}$ .

For proofs see [39] [40] [24]

Next we present a result on the ranks of $K_n(RG)$, $G_n(RG)$

**1.1.5. Theorem [39] [31]**

Let R be the ring of integers in a number field F, G a finite group; $\Gamma$ a maximal R-order containing $RG$ . Then for all n ≥ 1, rank $K_n(RG)$ =rank $G_n(RG)$ =rank $K_n(\Gamma)$.

For proof see [39] or [31]

Our next aim is to present a decomposition of $G_n(RG)$ n ≥ 0, G a finite Abelian group and extend these results to some non-Abelian groups e.g. quaternion and dihedral groups. First we have to develop some notations.

1.1.6. Let R be a left Noetherian ring and C a finite cyclic group of order n, generated by t, say. We write C = < t >. If $f: \mathbb{Z}C \to C$ is a ring homomorphism which is injective when restricted to C, then kerf is generated by $\Phi_n(t)$, the $n^{th}$ cyclotomic polynomial. Then the ideal $\Phi_n(t)\mathbb{Z}C$ is independent of the choice of generators.

Definie $R(C) = RC/\Phi_n(t)RC$ for any Noetherian ring R. Then $\mathbb{Z}(C)$ is an integral domain isomorphic to $\mathbb{Z}[\zeta_n]$ where $\zeta_n$ is the primitive $n^{th}$ root of 1. We identify $\mathbb{Q}(\zeta_n)$, the field of fractions of $\mathbb{Z}[\zeta_n]$ with $\mathbb{Q}(C)$. We write $R\langle C\rangle = R(C)(\frac{1}{n}) = R\left[\zeta_n, \frac{1}{n}\right]$. Note that $R(C) = R \otimes_{\mathbb{Z}} \mathbb{Z}(C)$ and $R\langle C\rangle = R \otimes \mathbb{Z}\langle C\rangle$

Now, Let G be a finite Abelian group, X (G) the set of cyclic quotients of G. Then, $\mathbb{Q}G \simeq \prod_{C\in X(G)} \mathbb{Q}(C)$. If $\Gamma$ is a maximal order in $\mathbb{Q}G$ containing $\mathbb{Z}G$, then $\Gamma \simeq \prod_{C\in X(G)} \mathbb{Z}(C)$. So, for any ring R, $R\otimes\Gamma = \prod_{c\in X(G)} R(C)$ and $R\otimes A = \prod_{C\in X(G)} R\langle C\rangle$

**1.1.7. Theorem [39] [77]**

Let G be a finite Abelian group R a Noetherian ring. Then, for all n ≥ 0,

$$G_n(RG) \simeq G_n(R\otimes A) \simeq \bigoplus_{C\in X(G)} G_n(R\langle C\rangle)$$

proof see [39] or [77].

1.1.8. Our next aim is to obtain some extensions of theorem 1.1.7 to some non-Abelian groups . e.g. Dihedral and Quaternion groups. We first give some definitions and fix notations.

Let R be a ring and G a group acting on R by ring automorphism. Call R a G-ring. The twisted groupring R # G is defined as the R-module $R\otimes\mathbb{Z}G$ with elements $a\otimes g$ (a ∈ R, g ∈ G) denoted a # g and multiplication defined by (a # g) · (a´ # g´) =ag (a´) # gg´. If G acts trivially on R, then R # G = RG. A sub ring G´ of G is cocyclic G/G´ is cyclic.

**1.1.9. Theorem [39] [77]**

Let $H = G \rtimes G_1$ be the semi-direct product of $G$ and $G_1$, where $G$ is a finite Abelian group and $G_1$ any finite group such that the action of $G_1$ on $G$ stabilizes every cocyclic subgroup of $G$, so that $G$ acts on each cyclic quotient C of $G$. Let R be a Noetherian ring. Then for all n ≥ 0, $G_n(RH) \simeq \bigoplus_{C\in X(G)} G_n(R\langle C\rangle \# G_1)$.

*Remarks.*This leads to the following result on the dihedral group. Note that $D_{2n} = G \rtimes G_1$ where $G$ is a cyclic group of order n and $G_1$ is a group of order 2.

*1.1.10. Theorem* Let $D_{2n}$ be the dihedral group of order $2n$ .Then

$G_n(\mathbb{Z}D_{2n}) = \bigoplus_{\substack{d|n \\ d\geq 2}} G_n(\mathbb{Z}[\zeta_d, \frac{1}{d}]_+) \oplus G_n(\mathbb{Z}[\frac{1}{2}])^{\varepsilon} \oplus G_*(\mathbb{Z})$ where $\varepsilon = \begin{cases} 1 \text{ if n is odd} \\ 2 \text{ if n is even} \end{cases}$ and

$\mathbb{Z}[\zeta_d, \frac{1}{d}]_+$ is the ring of integers in $\mathbb{Q}(\zeta_d)_+ = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$ the maximal subfield of the cyclotomic field $\mathbb{Q}(\zeta_d)$

1.1.11. We next consider the generalized quaternion group of order $4.2^s$. In general, let R be a commutative G-ring with identity (G a finite group), $c: G \times G \to R^*$ a normalized 2-cocycle with values in $R^*$ (see [76]). Then the crossed – product ring $R\#_c G$ is the R-module $R \otimes \mathbb{Z}G$ with multiplication given by $(a\#g)(a'\#g') = ag(a')c(g,g')\#gg'$. So $R\#_c G$ is an associative ring with identity $1\#1_G$. If c =1, then we obtain the twisted groupring. Now let H be a generalized quaternion group of order $4.2^s$.Then H has a presentation $H = \left\langle x, y | x^{2^s} = y^2, y^4 = 1, yxy^{-1} = x^{-1} \right\rangle$ Let $G_1 = \{1, \gamma\}$ be a two element group acting on $\mathbb{Q}[\zeta_{2^s}]$ by complex conjugation with fixed field $Q(\zeta_{2^s})_+ = \mathbb{Q}(\zeta_{2^s} + \zeta_{2^s})$ the maximal real subfield of $\mathbb{Q}(\zeta_{2^s})$. Let $c: G_1 \times G_1 \to \mathbb{Q}(\zeta_{2^{n+1}})^*$ be the normalized co cycle given by $c(\gamma, \gamma) = -1$,and let $\Sigma = Q(\zeta_{2^{s+1}})\#_c G_1$ be the crossed product algebra, and $\Gamma$ a maximal $\mathbb{Z}$ -order in $\Sigma$. Then, we have the following result

1.1.12. Theorem.

In the notation 1.1.11, we have

$$G_n(\mathbb{Z}H) \simeq \bigoplus_{j=0}^{s} G_*(\mathbb{Z}[\zeta_{2^s}, \frac{1}{2^s}]_+) \oplus G_*(\Gamma[\frac{1}{2^{s+1}}]) \oplus G_*(\mathbb{Z}[\frac{1}{2}])^2.$$

*Note:* It is still an open problem to obtain a decomposition for $G_n(\mathbb{Z}H)$ for an arbitrary finite group H.

1.1.13. Recall that if R is the ring of integers in a number field F, G a finite group, then the higher dimensional class group of RG is defined by $C\ell_n(RG) := Ker(K_n(RG) \to \bigoplus_{\underline{p}} K_n(\hat{R}_{\underline{p}}G)$ for all n ≥ 0, where $\hat{R}_{\underline{p}}$ is the completion of R at $\underline{p}$ and $\underline{p}$ runs through all the prime ideals of $R$.

Note that $C\ell_0(RG)$ coincides with the usual class group $C\ell(RG)$, which houses some topological/geometric invariants e.g. Swan-Wall invariants -Morever $C\ell_1(RG)$ is intimately connected with Whitehead torsion. It is also classical that $C\ell_0(RG)$, $C\ell_1(RG)$ are finite groups. We now present the following results

**1.1.14. Theorem [39] [18]**

(1) Let R be the ring of integers in a number field, G a finite group. Then $C\ell_n(RG)$ is a finite group for all n ≥ 1

(2) For all n ≥ 1, the only possible p-torsion in $C\ell_{2n-1}(RG)$ is for those primes p dividing the order of G.

For the proof of (1) see [39] or [28].
For the proof of (2) see [39] or [18].

**1.1.15. Remarks**

Observe that theorem 1.1.14 (2) above was stated for odd –dimensional class groups. Proving analogous result for even-dimensional class groups is still open. We also present the following result.

**1.1.16. Theorem. [39] [18]**

Let $S_r$ be the symmetric group of degree r and let r ≥ 0. Then $C\ell_{4n+1}(\mathbb{Z}S_r)$ is a finite 2-torsion group and the only possible odd torsion in $C\ell_{4n-1}(\mathbb{Z}S_r)$ that can occur are for those odd primes p such that $\frac{p-1}{2}$ divides n.

For a proof see [39] or [18].

***1.2. Consequences for Some Infinite Groups***

In this subsection, we indicate how results on $K_n(RG)$, $G_n(RG)$ could be extended to yield results on $K_n(RV), G_n(RV)$ where RV is a group ring of an infinite group V.

1.2.1. Let α be an automorphism of a ring A, with identity . We shall write $A_\alpha[T] = A_\alpha[t, t^{-1}]$ the α-twisted Laurent series ring over A. Here T= < t > is the infinite cyclic group generated by t. Note that additively $A_\alpha[T] = A[T] = A[t, t^{-1}]$ with multiplication given by $(at^i)\cdot(bt^j) = a\alpha^{-1}(b)t^{i+j}$ for a, b ∈ A.

Now let α be an automorphism of a finite group G, $R$ the ring of integers in a number field F. We also denote by α the automorphism induced on RG by α. Then $(RG)_\alpha(T) = RV$ where $V = G \underset{\alpha}{\times} T$ and the action of the infinite cyclic group T = < t > on G is given by $\alpha(g) = tgt^{-1}$ for all g ∈ G.V is called a virtually infinite cyclic group and K-theory of RV is fundamental to the Farrell – Jones conjecture which asserts that K-theory of an arbitrary discrete group should have as “building blocks” the K-theory of finite groups and virtually infinite cyclic groups

Note that an R-automorphism of RG extends to an F-auto morphism of FG, which we also denote by α. We now have the following result.

**1.2.2. Theorem [41]**

Let $V = G \underset{\alpha}{\times} T$ be a virtually infinite cyclic group where G is a finite group, α ∈ Aut(G) and the action of T on G is given by $\alpha(g) = tgt^{-1}$. Then

(1) $G_n(RV)$ is a finitely generated Abelian group for all n ≥ 1

(2) $\mathbb{Q}\otimes K_n(RV)\simeq\mathbb{Q}\otimes G_n(RV)\simeq\mathbb{Q}\otimes K_n(FV)$ for all n ≥ 2.

*Proof* : See [41]. Note that the proof of the theorem above like many others, is in the generality of replacing RG by an arbitrary R-order $\Lambda$ in a semi-simple F-algebra $\Sigma$. One then deduces above result for $\Lambda = RG, \Sigma = FG$.

We also have the following result which shows that $G_{2(m+1)}(\Lambda_\alpha[T])$ is actually finite for all odd positive integers m where F is a totally real field, when $\Lambda = RG$.

**1.2.3. Theorem [41]**

Let R be the ring of integers in a number field, G a finite group, T = < t > the infinite cyclic group and $V = G\underset{\alpha}{\times}T$ where α ∈ Aut (G) and the action of T on G is given by $\alpha(g) = tgt^{-1}$. Then for all odd positive integers m, $G_{2m+1}(RV)$ is a finite group when F is a totally real field.

Proof: See [41].

***1.3. Profinite higher K-theory of RG, RV***

1.3.1. Recall that in II, 1.3, we defined profinite K-theory of an exact category $\mathcal{C}$ by $K_n^{pr}(\mathcal{C},\hat{\mathbb{Z}}_\ell) := [M_{\ell^\infty}^{n+1}; \mathcal{B}Q\mathcal{C}]$ where $M_{\ell^\infty}^{n+1} = \underset{\overrightarrow{s}}{\lim} M_{\ell^s}^{n+1}$. We also defined $K_n(\mathcal{C},\hat{\mathbb{Z}}_\ell) := \underset{\overleftarrow{s}}{\lim} K_n(\mathcal{C},\mathbb{Z}/\ell^s)$ See page [39] [33]

Recall also that for any ring A with identity we write $K_n^{pr}(A,\hat{\mathbb{Z}}_\ell)$ for $K_n^{pr}(\mathcal{P}(A),\hat{\mathbb{Z}}_\ell)$ and for any Noetherian ring A we write $G_n^{pr}(A,\hat{\mathbb{Z}}_\ell)$ for $K_n^{pr}(\mathcal{M}(A),\hat{\mathbb{Z}}_\ell)$ We call $K_n^{pr}(A,\hat{\mathbb{Z}}_\ell)$ (resp $G_n^{pr}(A,\hat{\mathbb{Z}}_\ell)$ ) the profinite K-theory(resp. G – theory), of A. Similarly we write $K_n(A,\hat{\mathbb{Z}}_\ell)$ for $K_n(\mathcal{P}(A),\hat{\mathbb{Z}}_\ell)$ for and $G_n(A,\hat{\mathbb{Z}}_\ell)$ $K_n(\mathcal{M}(A),\hat{\mathbb{Z}}_\ell)$

We shall be interested in the cases A = RG and A = RV where R is a Dedekind domain (e.g. R=ring of integers in a number field or p-adic field F) and $V = G\underset{\alpha}{\times}T$ where α ∈ Aut (G) is given by $\alpha(g) = tgt^{-1}$ and T = < t > is the infinite cyclic group.

**1.3.2. Remarks**

Note that the profinite higher K-theory $K_n^{pr}(\mathcal{C},\hat{\mathbb{Z}}_\ell)$ for exact categories $\mathcal{C}$ (including equivariant exact categories) is a cohomology theory which generalizes classical profinite topological K-theory $K_0(\mathcal{C})\otimes\hat{\mathbb{Z}}_\ell$ where for a compact topological space X,

$\mathbb{C} = VB_{\mathbb{C}}(X)$ (resp $VB_{\mathbb{C}}^{G}(X)$) is the category of (finite dimensional) $\mathbb{C}$-vector bundles on X, ( resp. category of G – equivariant $\mathbb{C}$-vector bundles on a G – space X where G acts continuously on X) see [1]. The theory $K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_\ell)$ is also a K-theory analogue of classical continuous cohmology of schemes rooted in Arithmetic algebraic geometry. As such $K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_\ell)$ could be called continuous K-theory of $\mathbb{C}$. The following exact sequence provides a standard mechanism for computing $K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_\ell)$.

**1.3.3. Theorem [39] [33]**

If $\mathbb{C}$ is an exact category, $\ell$ a rational prime, then for all n ≥ 1, there exists an exact sequence $0 \to \varprojlim_s^1 K_{n+1}(\mathbb{C}, \mathbb{Z}/\ell^s) \to K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_\ell) \to K_n(\mathbb{C}, \hat{\mathbb{Z}}_\ell) \to 0$

Proof see [39] or [33].

**1.3.4. Remarks**

It follows from 1.3.3 that

(1) If A is any ring with identity we have exact sequences.

$$0 \to \varprojlim_s^1 K_{n+1}(A, \mathbb{Z}/\ell^s) \to K_n^{pr}(A, \hat{\mathbb{Z}}_\ell) \to K_n(A, \hat{\mathbb{Z}}_\ell) \to 0$$

(2) If A is a Noetherian ring , then we have an exact sequence

$$0 \to \varprojlim_s^1 G_{n+1}(A, \mathbb{Z}/\ell^s) \to G_n^{pr}(A, \hat{\mathbb{Z}}_\ell) \to G_n(A, \hat{\mathbb{Z}}_\ell) \to 0$$

In particular if R is the ring of integers in a number field or a p-adic field F,G a finite group, T = < t > an infinite cyclic group, $V = G \underset{\alpha}{\times} T$ , then A = RG and A=RV fit into the exact sequences in (1) and (2) above.

*1.3.5. Definition:* Let $\ell$ be a rational prime. An Abelian group G is said to be $\ell$-complete if $G \simeq \varprojlim_s (G/\ell^s G)$.

**1.3.6 Theorem [39] [33]**

Let $\mathbb{C}$ be an exact category such that $K_n(\mathbb{C})$ is a finitely generated Abelian group for all n ≥ 1. Let $\ell$ be a rational prime. Then $K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_\ell)$ is an $\ell$-complete profinite Abelian group for all n ≥ 2 Moreover, $K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_\ell) \simeq K_n(\mathbb{C}) \otimes \hat{\mathbb{Z}}_\ell \simeq K_n(\mathbb{C}, \hat{\mathbb{Z}}_\ell)$.

Proof: see [39] or [33].

**1.3.7. Corollary**

Let R be the ring of integers in a number field F, G a finite group.

Then $K_n^{pr}(RG,\hat{\mathbb{Z}}_\ell)$, $G_n^{pr}(RG,\hat{\mathbb{Z}}_\ell)$ are $\ell$-complete profinite Abelian groups for all n≥2. Moreover, $K_n^{pr}(RG,\hat{\mathbb{Z}}_\ell) \simeq K_n(RG)\otimes\hat{\mathbb{Z}}_\ell \simeq K_n(RG,\hat{\mathbb{Z}}_\ell)$ and $G_n^{pr}(RG,\hat{\mathbb{Z}}_\ell) \simeq G_n(RG)\otimes\hat{\mathbb{Z}}_\ell \simeq G_n(RG,\hat{\mathbb{Z}}_\ell)$

Proof: Follows from the fact that $K_n(RG)$, $G_n(RG)$ are finitely generated Abelian groups for all n ≥ 1 ( see 1.1.3 and 1.1.4).

**1.3.8. Corollary [40]**

Let R be the ring of integers in a number field F, G a finite group, $V = G\underset{\alpha}{\times}T$ where T = < t > is the infinite cyclic group.

Then $G_n^{pr}(RV,\hat{\mathbb{Z}}_\ell)$ is an $\ell$-complete profinite Abelian group for all n ≥ 2.

Moreover $G_n^{pr}(RV,\hat{\mathbb{Z}}_\ell) \simeq G_n(RV)\otimes\hat{\mathbb{Z}}_\ell \simeq G_n(RV,\hat{\mathbb{Z}}_\ell)$.

Proof: Follows since $G_n(RV)$ is finitely generated for all n ≥ 1 (See [40])

**1.3.9. Remarks**

When R is the ring of integers in a p-adic field, $K_n(RG)$, $G_n(RG)$ are no longer finitely generated. However, one can, through other techniques (see [39]) obtain the following $\ell$-completeness result for $G_n^{pr}(RV,\hat{\mathbb{Z}}_\ell)$.

**1.3.10. Theorem [39] [40]**

Let R be the ring of integers in a p-adic field F, G a finite group. Then for all n ≥ 2, $G_n^{pr}(RV,\hat{\mathbb{Z}}_\ell)$ is an $\ell$-complete profinite Abelian group.

1.3.11. Before stating the next result ( 1.3.12), we first explain the construction of the map $\varphi$ in the theorem.

Note that for any exact category $\mathcal{C}$, the natural map $M_{\ell^\infty}^{n+1} \to S^{n+1}$ induces a map $[S^{n+1},\mathcal{B}Q\mathcal{C}] \xrightarrow{\varphi} [M_{\ell^\infty}^{n+1},\mathcal{B}Q\mathcal{C}]$ i.e. $K_n(\mathcal{C}) \xrightarrow{\varphi} K_n^{pr}(\mathcal{C},\hat{\mathbb{Z}}_\ell)$. So when $\mathcal{C}$ = $\mathcal{M}$(RV), where $V = G\underset{\alpha}{\times}T$, G a finite group, T = < t > the infinite cyclic group and α ∈ Aut(G) given by $\alpha(g) = tgt^{-1}$, we obtain a map φ; $G_n(RV) \to G_n^{pr}(RG,\hat{\mathbb{Z}}_\ell)$

If B is an Abelian group, we write div B for the subgroup of divisible element of B.
We now state the following result.

**1.3.12. Theorem [40]**

Let R be the ring of integer in a number field F, G a finite group $V = G \times_{\alpha} T$ where T = < t > is the infinite cyclic group and α ∈ Aut(G) is given by $\alpha(g) = tgt^{-1}$, RV the group ring of V over R. Then for all n ≥ 2

(i) div $G_n^{pr}(RV, \hat{\mathbb{Z}}_{\ell})$ =0

(ii) $G_n^{pr}(RV, \hat{\mathbb{Z}}_{\ell}) \simeq G_n(RV, \hat{\mathbb{Z}}_{\ell})$ is an $\ell$ -complete profinite Abelian group

(iii) The map $\varphi : G_n(RV) \to G_n^{pr}(RV, \hat{\mathbb{Z}}_{\ell})$ is injective with uniquely $\ell$ -divisible cokernel

## Section 2. Some Results on the Actions of Algebraic Groups

### *2.1. The representation ring R (H) and the group* $K_0(VB_G(G/H))$

2.1.1. Let F be a field, G an algebraic F-group, $\mathcal{P}(F)_G$ the category of representations of G in $\mathcal{P}$ (F) where $\mathcal{P}$ (F) is the category of finite-dimensional vector spaces over F. We shall also denote this category by $\mathrm{Rep}_F(G)$. Note that $\mathrm{Rep}_F(G)$ is an exact category and we denote $K_0(\mathrm{Rep}_F(G))$ by $R_F(G)$ or R (G,F) or just R(G) when the context is clear .

Note that R(G) is a free Abelian group generated by the classes of irreducible representations and that R(G) also has a ring structure induced by tensor product. Call R(G) the representation ring. (see [50] [55])

Since $\mathrm{Rep}_F(G)$ is an exact category, $K_n(\mathrm{Rep}_F(G))$ is defined for all n ≥ 0 and we denote $K_n(\mathrm{Rep}_F(G))$ by $G_n(G,F)$.

2.1.2. For any G-scheme X, let $\mathcal{VB}_G(X)$ be the exact category of G-equivariant (algebraic) vector bundles on X. We saw in I, 3.2.2 that if H is a subgroup of G and X = G/H the homogeneous G-space, then we have an equivalence of categories:

$\mathrm{Rep}_F(G) \underset{res}{\overset{ind}{\rightleftarrows}} \mathcal{VB}_G(G/H)$ where the maps ind and res were described in I, 3.2.2.

So, we have an Abelian group isomorphism $R(H) \simeq K_0\mathcal{VB}_G(G/H)$. We shall denote $K_0(\mathcal{VB}_G(G/H))$ by $K_0^G(G/H)$

2.1.3. Let $\alpha : H \to G$ be a homomorphism of algebraic group; then f induces a ring homomorphism $\alpha^* : R(G) \to RH$. Hence we can consider $R(H)$ as an $R(G)$ - module. If f is an embedding of a subgroup H into G, call $\alpha^*$ a restriction map.

If E is a field extension of F and we write $G_E$ for $G(E) = G \times_F E$, then the exact functor $\mathrm{Rep}(G) \to \mathrm{Rep}(G_E) : [V] \to [V \otimes_F E]$ induces a ring homomorphism. If E/F is a finite extension, we also have an exact functor $\mathrm{Rep}(G_E) \to \mathrm{Rep}(G)$ which takes a $G_E$ -module M over E to itself considered as a G-module over F. This induces a ‘corestriction 'map: $R(G_E) \to \mathrm{R(G)}$. Note that the composition $R(G) \to \mathrm{R(G_E)} \to \mathrm{R(G)}$ coincides with multiplication by [E, F]. In particular the homomorphism $R(G) \to R(G_E)$ is injective (see [50]).

### *2.2. The groups $K_n(G, X), G_n(G, X)$, X a G-Scheme*

2.2.1. Let G be an F-group and X a G-scheme. In I, 3.3.2 we defined equivariant exact categories $\mathcal{M}$(G, X), $\mathcal{P}$(G, X). Let $K_n(G, X)$ denote $K_n(\mathcal{P}(G, X)), G_n(G, X)$ .denote $K_n(\mathcal{M}(G), X))$.

- Note that if X = $Spec(F)$, and n = 0 then $R(G) = G_0(G, Spec(F) = K_0(G, Spec(F))$
- When X = G/H is a homogeneous space where H is subgroup of G, $R(H) \simeq G_0(G, G/H) \simeq K_0(G, G/H) \simeq K_0^G(G/H)$ (see 2.1)
- If U is a unipotent subgroup of G, then, there is a natural bijection between the sets of irreducible representation of G and irreducible representations of G/U. Hence the natural map $R(G/U) \to R(U)$ is an isomorphism. (See [50])
- $G_n(G, -)$ is contravariant with respect that G-morphism and is covariant with respect to projective G-morphisms of G-schemes (See [50])
- $K_n(G, -)$ is contravariant with respect to G-morphisms of G-varieties. ( see [50])
- The inclusion of categories $\mathcal{P}(G, X) \to \mathcal{M}(G, X)$ induces a homomorphism $K_n(G, -) \to G_n(G, -)$
- A morphism $\alpha : G \to H$ induces exact functors $\mathcal{M}(G, X) \to \mathcal{M}(H, X)$, $\mathcal{P}(G, X) \to \mathcal{P}(H, X)$ and hence group homomorphism $G_n(G, X) \to G_n(H, X)$, $K_n(G, X) \to K_n(H, X)$

**2.2.2**. The restriction functor $\mathcal{M}(G, X) \xrightarrow{f} \mathcal{M}(X)$ induces a group homorphism $G_n(G, X) \xrightarrow{f_*} G_n(X)$. The next few results provide information on the map $f_*$, which links equivariant K-theory to ordinary K-theory of X. Note that $\mathcal{P}(G, X) \to \mathcal{P}(X)$ also induces $K_n(G, X) \to K_n(X)$.

**2.2.3. Theorem [50]**

If $G_0(G, X) \to G_0(X)$ surjective, then $\mathrm{Pic(G_E)} = 0$ for any finite extension E/F. If $\mathrm{Pic(G_E)} = 0$ and X is affine, then $G_0(G, X) \to G_0(X)$ is surjective

proof; see [50].

**2.2.4. Theorem [50]**

Let G be an algebraic F-group. For all n ≥ 0 $G_n(G, X) \to G_n(X)$ is a split surjection if X is a smooth projective variety.

Proof see [50]

**2.2.5. Theorem [50]**

Let G be a split reductive F-group. Then for any G-scheme X, the homomorphism $G_0(G,X) \to G_0(X)$ induces an isomorphism $\mathbb{Z} \otimes_{R(G)} G_0(G,X) \xrightarrow{\sim} G_0(X)$.

Proof: See [50].

**2.2.6. Theorem [50]**

Let G be a split reductive group, X a smooth projective variety. Then the homomorphism $K_n(G,X) \to K_n(X)$ induces an isomorphism $\mathbb{Z} \otimes_{R(G)} K_n(G,X) \simeq K_n(X)$

Proof See [50].

**2.2.7. Theorem [50]**

Let U be a split unipotent group, X a U -scheme. Then the restriction homomorphism $G_n(U,X) \to G_n(X)$ is an isomorphism.

**2.2.8.** Recall from 2.1.3 and 2.2.1 that a group homomorphism $\alpha : H \to G$ induces a ring homomorphism $\alpha^* : R(G) \to RH$ as well as group homomorphism $G_n(G,X) \to G_n(H,X)$ and $K_n(G,X) \to K_n(H,X)$. The next few results present some information on $G_n(G,X) \to G_n(H,X)$.

**2.2.9. Theorem [50]**

Let G be an F-group and H a closed subgroup of G such that the factor – variety G/H is isomorphic to $A_F^1$. Then for any G-scheme X, the restriction map $G_n(G,X) \to G_n(H,X)$ is an isomorphism for all n ≥ 0.

Proof See [50].

**2.2.10. Theorem [50]**

Let G be a split solvable group, T ⊂ G a split maximal torus, X a G – scheme. Then the restriction homomorphism $G_n(G,X) \to G_n(T,X)$ is an isomorphism.

## *2.3. Higher K-theory of Twisted Flag Varieties*

2.3.1 Recall from I, 3.2.3, that if $\tilde{G}$ is a semi-simple, connected and simply connected F-split algebraic group over a field F, and $\tilde{P} \subset \tilde{G}$ a parabolic subgroup of $\tilde{G}$ containing the maximal torus $\tilde{T}$ .then $\mathcal{F} = \tilde{G}/\tilde{P}$ is a flag variety We also saw in II, 1.3.11 that $R(\tilde{P})$ is a free $R(\tilde{G})$ -module of rank $s(\mathcal{F})$ where $s(\mathcal{F}) = [W; W_{\tilde{P}}]$, W = the Weyl group of $\tilde{G}$,

and $W_{\tilde{P}} = \{w \in W| \, w\tilde{P}w^{-1} = \tilde{P}\}$.

Now let ${}_c\mathcal{F}$ be the twisted flag variety relative to the co cycle

$c : Gal(F_{sep} / F) \to \tilde{G}(F_{sep})$ (see II 1.3.13 and [55] [40]).

Now for any $\tilde{V} \in \mathcal{VB}_G(\mathcal{F})$ (see II, 1.3.8), let ${}_c\tilde{V}$ be the vector bundle over ${}_c\mathcal{F}$ obtained by twisting $\tilde{V}$ by c. Then we have a biexact functor $\mathrm{Rep}_F(\tilde{P}) \times \mathcal{P}(F) \to \mathcal{VB}_{\tilde{P}}({}_c\mathcal{F}) : (V, M) \to {}_c\tilde{V} \otimes_F M$ which induces a pairing $\mu_c : R_F(\tilde{P}) \underset{\mathbb{Z}}{\otimes} K_n(F) \to K_n({}_c\mathcal{F})$. We now have the following result.

**2.3.2. Theorem [55]**

In the notation of 2.3.1, we have

(1) $\mu_c : R_F(\tilde{P}) \underset{\mathbb{Z}}{\otimes} K_n(F) \to K_n({}_c\mathcal{F})$ is surjective for all n ≥ 0.

(2) $\mu_c$ induces a graded ring isomorphism $R_F(\tilde{P}) \underset{R_F(\tilde{G})}{\otimes} K_*(F) \xrightarrow{\sim} K_*({}_c\mathcal{F})$

(3) $R_F(\tilde{P}) \simeq R_F(\tilde{G})^{s(\mathcal{F})}$ where $R_F(\tilde{P})$ is considered as an $R_F(\tilde{G})$ - module by restriction of representatives.

(4) If $\{a_i| \, i = 1, 2...s\}$ is a free $R_F(\tilde{G})$ -basis of $R_F(\tilde{P})$ then $\bigoplus_{i=1}^{s} K_n(F) \simeq K_n({}_c\mathcal{F})$ is an isomorphism.

Proof See [55]

**2.3.3. Theorem [40]**

In the notation of 2.3.1 and 2.3.2, let F be a number field. Then for all n ≥ 1,

1) $K_{2n+1}({}_c\mathcal{F}, B)$ is a finitely generated Abelian group.

2) $K_{2n}({}_c\mathcal{F}, B)$ is a torsion group and has no non-trivial divisible subgroups

Proof: see [40].

We next present the following result on the local structure.

**2.3.5. Theorem [40]**

Let F be a p-adic field, $\ell$ a rational prime such that $\ell \neq p$. Then for all n ≥ 1, and any separable F-algebra B, $K_n({}_c\mathcal{F}, B)_\ell$ is a finite group.

Proof : See [40]

### *2.4. Finiteness Results for Some Objects of the Motivic Category $\mathbb{C}$(G)*

2.4.1. Let G be an algebraic group over a field F. By considering a smooth projective G – scheme as an object of a category $\mathbb{C}$ (G) defined below, we have similar finiteness results to those for $K_n({}_cX, B)$ where c is a 1 co-cycle, ${}_cX$ is the c-twisted form of X and B is a separable F-algebra.

2.4.2. The category $\mathbb{C}$ (G) is constructed as follows (the construction is due to I. Panin see [55], or [40]. The objects of $\mathbb{C}$ (G) are pairs (X, A) where X is a smooth projective G – scheme and A is a finite dimensional separable F – algebra on which G acts by F – algebra automorphisms. Define $Hom_{\mathbb{C}(G)}((X,A),(Y,B)) := K_0(G, X \times Y, A^{op} \otimes_F B)$.
Composition of morphisms is defined as follows: if
$u:(X,A) \to (Y,B)$, $v:(Y,B) \to (Z,C)$ are two morphisms, then the composite is defined by $v \circ u := p_{13}^*(p_{23}^*(v) \otimes_B p_{12}^*(u))$ where $p_{12} : X \otimes Y \otimes Z \to X \otimes Y$,
$p_{13} : X \otimes Y \otimes Z \to X \otimes Z$ ,and $p_{23} : X \otimes Y \otimes Z \to Y \otimes Z$. The identity endomorphism of (X, A) in $\mathbb{C}$ (G) is the class $[A \otimes_F O_\Delta]$ (where $\Delta \subset X \times X$ is the diagonal) in
$K_0(G, X \times X, A^{(\gamma)} \otimes_F A) = \mathrm{End}_{\mathbb{C}(G)}(X,A)$. We now have the following results.

**2.4.3. Theorem. [40]**

Let $\alpha : C \xrightarrow{\sim} (X,F)$ be an isomorphism in the category $\mathbb{C}$(G),
i.e., $\alpha : (Spec(F), C) \xrightarrow{\sim} (X,F)$. For every 1 – cocycle: $Gal(F_{sep}/F) \to G_{F\,sep}$ and any finite dimensional separable F-algebra B, let $K_n({}_cY, B)$ be as defined in II, 1.2.3.

a) If F is a number field, then for n ≥ 1,

(i) $K_{2n+1}({}_cX, B)$ is a finitely generated Abelian group.

(ii) $K_{2n}({}_cX, B)$ is a torsion group and has no non-trivial divisible elements.

b) If F is a p-adic field, $l$ a rational prime such that $l \neq p$ ,then for all n ≥ 1 and any separable F-algebra B, $K_n({}_cX, B)_l$ is a finite group.

Proof : See [40]

### *2.5. Profinite Higher K-Theory of Twisted Flag Varieties*

In this subsection, we obtain some $l$-completeness and other results for some twisted flag varieties as well as Brauer- Severi varieties over number fields and p-adic fields. Recall that if $l$ is a rational prime an Abelian group H is said to be $l$-complete if $H = \varprojlim_s H / l^s H$. Recall the definition of profinite K-theory $K_n^{pr}({}_c\mathcal{F}, B), \mathbb{Z}_\ell)$ from II, 1.5.2 (vii)

**2.5.1. Theorem [40]**

Let F be a number field, $\tilde{G}$ a semi-simple, connected, simply connected split algebraic group over $F, \tilde{P}$ a parabolic subgroup of $\tilde{G}$, B a finite dimensional separable F-algebra. Then for all n ≥ 1,

(1) $K_n^{pr}((_c\mathcal{F}, B), \mathbb{Z}_\ell)$ is an $l$-complete profinite Abelian group.

(2) $divK_n^{pr}((_c\mathcal{F}, B), \mathbb{Z}_\ell) = 0$

Proof see [40]

**2.5.2. Remarks**

The following results can be proved by procedures similar to those used to prove the result above. See [40] for details.

(1) If F a number field, then $K_{2n}^{pr}(_\gamma\mathcal{F}, \hat{\mathbb{Z}}_\ell) = 0$.

(2) If V is a Brauer-Severi variety over a number field F, then for all n ≥ 2, $K_{2n}^{pr}(V, \hat{\mathbb{Z}}_\ell)$ is $l$-complete and $divK_{2n}^{pr}(V, \hat{\mathbb{Z}}_\ell) = 0$

**2.5.3**. Our next aim is to consider the situation when F is a p-adic field. Before doing this, we make some general observations. Note that for any exact category $\mathbb{C}$, the natural map $M_{l^\infty}^{n+1} \to S^{n+1}$ induces a map $[S^{n+1}, BQC] \xrightarrow{\varphi} [M_{l^\infty}^{n+1}, BQC]$.

i.e., (I) $K_n(\mathbb{C}) \xrightarrow{\varphi} K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_l)$ and hence maps

(II) $K_n(\mathbb{C})/l^s \to K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_l)/l^s$ and

(III) $K_n(\mathbb{C})/[l^s] \to K_n^{pr}(\mathbb{C}, \hat{\mathbb{Z}}_l)[l^s]$

We denote the maps in (II) and (III) also by $\varphi$ by abuse of notation. We now present the following result.

**2.5.4. Theorem. [40]**

Let p be a rational prime, F a p-adic field, $\tilde{G}$ a semi-simple connected and simply connected split algebraic group over $F, \tilde{P}$ a parabolic subgroup of $\tilde{G}$, c a 1-cocycle $Gal(F_{sep}/F) \to \tilde{G}(F_{sep})$, $_c\mathcal{F}$ the c-twisted form of $\mathcal{F}$, B a finite dimensional separable F-algebra, $l$ a rational prime such that $l \neq p$. Then for all n ≥ 2.

(a) $K_n^{pr}((_c\mathcal{F}, B); \hat{\mathbb{Z}}_l))$ is an $l$-complete profinite Abelian group.

(b) $K_n^{pr}((_c\mathcal{F}, B), \mathbb{Z}_\ell) \simeq K_n((_c\mathcal{F}, B); \hat{\mathbb{Z}}_l)$.

(c) The map $\varphi$: $K_n(_c\mathcal{F}, B) \to K_n^{pr}((_c\mathcal{F}, B); \hat{\mathbb{Z}}_l)$ induces isomorphisms

(1) $K_n(_c\mathcal{F}, B)[l^s] \simeq K_n^{pr}((_c\mathcal{F}, B); \hat{\mathbb{Z}}_l)[l^s]$

(2) $K_n({}_c\mathcal{F}, B)/l^s \simeq K_n^{pr}(({}_c\mathcal{F}, B); \hat{\mathbb{Z}}_l)/l^s$.

(d) Kernel and cokernel of $K_n({}_c\mathcal{F}, B) \to K_n^{pr}(({}_c\mathcal{F}, B); \hat{\mathbb{Z}}_l)$ are uniquely $l$-divisible.

(e) $div K_n^{pr}(({}_c\mathcal{F}, B); \hat{\mathbb{Z}}_l) = 0$ for n ≥ 2

Proof : See [40]

**2.5.5. Remarks**

(a) Let V be a Brauer-Severi variety over a p-adic field F. By a similar proof to that of 2.5.4 we have

(i) $K_n^{pr}(V, \hat{\mathbb{Z}}_l) \simeq K_n(V, \hat{\mathbb{Z}}_l)$ is an $l$-complete profinite Abelian group.

(ii) $K_n(V)/l^s \simeq K_n^{pr}(V, \hat{\mathbb{Z}}_l)/l^s$ and $K_n(V)[l^s] \simeq K_n^{pr}(V, \hat{\mathbb{Z}}_l)[l^s]$.

Kernel and cokernel of $K_n(V) \to K_n^{pr}(V, \hat{\mathbb{Z}}_l)$ are uniquely $l$-divisible.

$div K_n^{pr}(V, \hat{\mathbb{Z}}_l) = 0$

b) Finally, if ${}_cX$ is as in 2.4.3, we have similar results to those of 2.5.4 for $K_n^{pr}({}_cX, B)$, etc.

# REFERENCES


[1] M.F. Atiyah, *K-theory*. W.A. Benjamin (1967).

[2] H. Bass *Algebraic K-theory* W.A. Benjamin (1968).

[3] H. Bass, Lenstra's Calculation of $G_0(R\pi)$ and Application to Morse-Smale Diffeomophisms. *Lecture Notes in Math 88*, Springer, 1981, 287 -291.

[4] H . Bass, A.O. Kuku and C. Pedrini (eds), Algebraic K-theory and its Applications. ICTP K-theory Proceedings. *World Scientific* (1999).

[5] W. Browder, Algebraic K-theory with Coefficients $\mathbb{Z}/P$ . *Lecture Noes in Math 657*. Springer. 40 - 84.

[6] C.W. Curtis and I. Reiner, *Methods of Representation theory I and II.* Wiley (1982); (1987).

[7] A.W.M. Dress, Contributions to the Theory of Induced Representations. *Lecture Notes in Math.* Springer 1973,183 - 240.

[8] A.W.M. Dress, Induction and Structure Theorems for Orthogonal Representations of Finite Groups, *Ann. Math* 102 (1975) 291-325.

[9] A.W.M. Dress and A.O. Kuku, The Cartan Map for Equivariant Higher K-groups. *Comm. in Algebra*, 9 (1981), 727 -747.

[10] A.W.M. Dress and A.O. Kuku, A Convenient Setting for Equivariant Higher, Algebraic K-theory. *Lecture Notes in Math*. 966, Springer (1986) 58 - 68.

[11] J.A.Green. Axiomatic representation theory of finite groups. *J. Pure Appl. Algebra* 1(1)(97), 41 – 72.
[12] X. Guo and A.O. Kuku, Higher Class Groups of Generalized Eichler Orders.*Comm. In Algebra* 33 (2005), 709 -718.
[13] X. Guo and A.O. Kuku, Wild Kernels for Higher K-theory of Division and Semi-simple Algebras, Beitrage ZÜr *Algebra und Geometrie*, 47 (2006) (1), 1 – 14.
[14] R.Hartshone, *Algebraic Geometry*. Springer, NY 1977.
[15] A Hatcher and J.Wagoner. Pseudo – isotopies on compact manifolds. Asterisq 6 (1973).
[16] M. Karoubi, *K-theory : An Introduction*. Springer 1978.
[17] M. Karoubi, A.O. Kuku and C. Pedrini (eds), Contemporary Development in Algebraic K-theory. *ICTP Lecture Notes Series*, 15 (2003).
[18] M.Kolster and R.C. Laubenbacher. Higher class groups of orders, *Math. Z.* 228 (1998) 229 – 246.
[19] A.O. Kuku. Some Algebraic K-theory Application of LF and NF Functors. *Proc. AMS* (37) (2) 1973, 36 - 365.
[20] A.O. Kuku, Whitehead Group of Orders in p-adic Semi-simple Algebras. *J. Algebra 25* (1973), 415 - 418.
[21] A.O. Kuku, Some Finiteness Theorems in the K-theory of Orders in p-adic Algebras. *J. Lond. Math. Soc* (13) (1) 1976,122 - 128.
[22] A.O. Kuku, $SK_n$ of Orders and $G_n$ of Finite Rings, *Lect. Notes in Math. 51*, Springer-Verlag (1976) 60 - 68.
[23] A.O. Kuku, $SG_n$ of Orders and Group Rings, *Math. Z.* 165 (1979) 291 – 295.
[24] A.O. Kuku, Higher Algebraic K-theory of Group Rings and Orders in Algebras over number fields, *Comm. Algebra* 10(8) (1982) 805 - 816.
[25] A.O Kuku, Equivariant K-theory and the cohomology of profinite groups, *Lect. Notes in Math.* 1046 (1984), Springer – Verlag, 234 – 244.
[26] A.O. Kuku, K-theory of group rings of finite groups over maximal orders in division algebras, *J. Algebra* 91 (1) 1984) 18 - 31.
[27] A.O. Kuku, Axiomatic theory of induced representations of finite groups, *Les cours du CIMPA*, No. 5, Nice, France, 1985.
[28] A.O. Kuku, $K_n, SK_n$ , of integral group rings and orders, *Contemp. Math. AMS* 55 (1986) 333 - 338.
[29] A.O. Kuku, Some finiteness results in the higher K-theory of orders and grouprings, *Topology Appl*. 25 (1987) 185 – 191.
[30] A.O. Kuku, Higher K-theory of modules over EI categories, *Africa Mat.3* (1996) 15 - 27.
[31] A.O. Kuku, Ranks of $K_n$ and $G_n$ of orders and group rings of finite groups over integers in number fields, *J. Pure Appl. Algebra* 138 (1999),39 - 44.
[32] A.O. Kuku, Equivariant higher K-theory for compact Lie-group actions. *Bietrage zÜr Algebra and Geometric* 41 (1) (2000), 141 - 150.
[33] A.O. Kuku, Profinite and continuous higher K-theory of exact categories, orders and group-rings, *K-theory* 22 (2001)367 -392.

[34] A.O. Kuku, Classical Algebraic K-theory: the functors $K_0, K_1, K_2$. *Handbook of Algebra 3* (2003) Elsevier, 157 - 196.
[35] A.O. Kuku, K-theory and Representation Theory - Contemporary developments in Algebraic K-theory, *ICTP Lect. Series*, No. 15 (2003) 259 -356.
[36] A.O. Kuku, Higher Algebraic K-theory, *Handbook of Algebra*, 4 (2006), Elsevier, 3 - 74.
[37] A.O. Kuku, Finiteness of higher K-groups of orders and group-rings. *K-theory* (2005), (36) 51 - 58.
[38] A.O. Kuku, Equivariant higher algebraic K-theory for Waldhausen categories, *Bietage ZÜr Algebra und Geometridc contributions to Algebra and Geometry.* vol.47, (2) 583 - 601 (2006).
[39] A.O. Kuku, *Representation theory and higher Algebraic K-theory*. Chapman and Hall (2007).
[40] A.O. Kuku, Profinite equivariant higher algebraic K-theory for action of algebraic Groups. *Homology, Homotopy and Applications* (To appears).
[41] A.O. Kuku, Higher Algebraic K-theory for twisted Laurent series rings over orders and semi-simple algebras. *Algebras and Representation theory*. (2008)11:355-368.
[42] A.O. Kuku and G.Tang, Higher K-theory of group-rings of virtually infinite cyclic groups. *Math. Ann 322* (2003) 711 -725.
[43] A.O. Kuku and M. Mahdavi-Hezeveli. Subgroups of $GL_n(R)$ for local rings R. *Comm. In Algebra* 32 (15) 2004 1805 – 1902.
[44] A.O. Kuku and G. Tang, An explicit computation of "bar" homology groups of a non-unital ring. *Beitrage ZÜR Algebra und Geometrie* 44 (2) 2003,375 - 382.
[45] T.Y. Lam, Induction techniques for Grothendieck groups and Whitehead group of finite groups. *Ann. Sc. Ecole. Norm Sup.* Paris 1 (1968) 91 - 148.
[46] T.Y. Lam and I.Reiner, Relative Grothendieck groups. *J. Algebra* 11 (1969) 213-242.
[47] R.C. Laubenbacher and D. Webb. On $SG_n$ of orders. *J. Algebra* 133 (1990) 125-131.
[48] H.W. Lenstra, Grothendieck groups of Abalian group-rings. *J. Pure. App. Alg*. 20 (1981) 173-193.
[49] S. Maclane categories for the working mathematician, Springer – verlag (1971).
[50] A.S. Merkurjev. Comparison of equivariant and ordinary K – theory of Algebraic varieties. *St Petersburg math. J*. 9 (1998) No 4, 815 – 850.
[51] J. Minor, Whitehead torsion. *Bull. Amer. Math Soc*. (72) 1966, 358 – 426.
[52] J. Milnor, *Introduction to Algebraic K-theory*. Princeton (1971).
[53] Neisendorfer, Primary Homotopy theory. *Memoir Amer. Math*. Soc. 232 AMS (1980).
[54] R. Oliver, *Whitehead groups of finite groups*. Cambridge Univ. Press (1988).
[55] I – Panin. *On the Algebraic K – theory of twisted flag varieties K – theory* 8 (1994) 541-585.
[56] D. Quillen, On the cohomology and K – theory of the general linear groups of a finite field. *Ann Math* 96 (1972) 552 – 586.
[57] D. Quillen, Higher algebraic K-theory. *Lecture Notes in Math.* Springer-Verlag (1973) 85 - 147.
[58] D. Quillen, Finite generation of the K-groups of algebraic integers. *Lecture Notes in Math.* 341 Springer-Verlag (1973), 195 -214.

[59] G.Segal, Equivariant K-theory. *Publ. Math* IHES, 34 (1968).

[60] G. Segal, Representation ring of compact Lie-groups, *IHES* (34) 1968, 113 - 128.

[61] J.P. Serre, *Linear representations of finite groups.* Springer-Verlag, (1977).

[62] J.P. Serre, *Local fields*. Springer-Verlag, Berlin (1979).

[63] E.A. Spanier, *Algebraic topology*. Mcgraw- Hill (1966).

[64] T.A Springer Linear Algebraic Groups. Second edition, Birkhauser Boston 1998.

[65] V. Srinivas, *Algebraic K-theory*. Progress in Math 90.

[66] A.A. Suslin, Stability in Algebraic K-theory. *Lect. Notes in Math*. 966 Springer-Verlag (1982) 304 - 333.

[67] A.A. Suslin and A.V. Yufsyakov, K-theory of local division algebras. *Soviet Math. Docklady* 33 (1986) 794 -798.

[68] A.A. Suslin and M. Wodziki, Excision in algebraic K-theory. *Ann. Math* (2) 136 (1) 1992 51-122.

[69] R.G. Swan, Vector bundles and projective modules. *Trans AMS* 105 (1962) 264 – 277.

[70] R.G.Swan, *K-theory of finite Groups and orders Springer – Verlag*, lecture notes 149 (1979)

[71] R.G. Swan, Algebraic K-theory. *Lecture Notes in Math* 76 Springer-Verlag (1968).

[72] R.W. Thomason, Algebraic K-theory of group scheme actions. In *algebraic topology and algebraic K-theory*. Proceedings Princeton, NJ (1987) 539 – 563.

[73] J.B. Wagoner, Continuous cohomology and p-adic K-theory. *Lecture Notes in Math* 551, Springer-Verlag 241-248.

[74] C.T.C. Wall, Finiteness conditions for CW-complexes. *Ann Math*. 81 (1965) 56-69.

[75] C.T.C. Wall, Norms of units of group rings. *Proc. Lond. Math. Soc* (1974) 593-632.

[76] D. Webb, Grothendieck groups of dihedral and quaternion groups. *J. Pure Appl. Alge*. (35) (1985) 197 -223.

[77] D. Webb, The Lestra map on classifying spaces and G-theory. *Invent. Math*. 84 (1980) 73 - 89.

[78] G.W. Whitehead, *Elements of homotopy theory*. Springer-Verlag, NY 1978.

[79] J.H.C. Whitehead, Simple homotopy types. *Am. J. Math* 72 (1950.) 1 - 57.