Afr. Mat. (2011) 22:91–104
DOI 10.1007/s13370-011-0013-2

Profinite higher algebraic K-theory of twisted flag varieties

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Received: 16 July 2010 / Accepted: 14 January 2011 / Published online: 17 February 2011

Abstract Let G be an algebraic group over a field F. In this paper, we study profinite equivariant higher algebraic K-theory for the actions of G leading to computations of profinite higher algebraic K-theory of twisted flag varieties. More precisely, let ${}_{\gamma}\mathcal{F}$ be a twisted flag variety (see 1.2.3), B a finite dimensional separable F-algebra, and ℓ an odd rational prime. When F is a number field, we prove that for all $n \geq 1$, $K_{2n}^{\text{pr}}(({}_{\gamma}\mathcal{F}, B), \hat{Z}_{\ell})$ is an ℓ-complete Abelian group and div $K_{2n}^{\text{pr}}(({}_{\gamma}\mathcal{F}, B), \hat{Z}_{\ell}) = 0$. If F is a p-adic field, we prove that for $n \geq 1$ $K_n^{\text{pr}}(({}_{\gamma}\mathcal{F}, B), \hat{Z}_{\ell}) \approx K_n(({}_{\gamma}\mathcal{F}, B), \hat{Z}_{\ell})$ are ℓ-complete profinite Abelian groups and div $K_n^{\text{pr}}(({}_{\gamma}\mathcal{F}, B), \hat{Z}_{\ell}) = 0$. As preliminary results, we prove that when F is a number field, then for the ordinary higher K-theories, we have $K_{2n+1}({}_{\gamma}\mathcal{F}, B)$ are finitely generated Abelian group and $K_{2n}({}_{\gamma}\mathcal{F}, B)$ are torsion while if F is a p-adic field, then for all $n \geq 2$, $K_n({}_{\gamma}\mathcal{F}, B)_{\ell}$ are finite groups.

Keywords Algebraic groups · Group scheme · Twisted flag varieties · Profinite higher algebraic K-theory · Separable algebras over number fields and p-adic fields · ℓ-complete

Mathematics Subject Classification (2000) 14L10 · 14L30 · 16S34 · 16S35 · 16H10 · 19D06 · 19D50

Introduction

Let G be an algebraic group over a field F. The aim of this paper is to study profinite equivariant higher algebraic K-theory for G-actions with the goal of computing profinite higher K-theory of twisted flag varieties.

As a preliminary step, we briefly review equivariant higher K-theory for schemes in the style of Thomason [19], which translates into ordinary higher K-theory of flag varieties. In the process, we prove preliminary results on ordinary higher K-theory of twisted flag varieties through the result of Panin which reduces the study of ordinary higher K-theory of twisted

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flag varieties to ordinary higher K-theory of finite-dimensional semi-simple algebras. (see Theorem 1.2.6 or [13]).

More precisely, let $\tilde{G}$ be a semi-simple connected and simply connected F-split algebraic group over a field F, $\tilde{P}$ a parabolic subgroup of $\tilde{G}$, $\mathcal{F} = \tilde{G}/\tilde{P}$, ${}_\gamma\mathcal{F}$ the twisted form of $\mathcal{F}$ with respect to the 1-cocycle $\gamma : \mathrm{Gal}(F_{sep}/F) \to G(F_{\mathrm{sep}})$ (see 1.2 or [13]), B a finite-dimensional separable F-algebra and $K_n({}_\gamma\mathcal{F}, B)$ Quillen K-theory of the category $\mathcal{VB}_{\tilde{G}}({}_\gamma\mathcal{F}, B)$ of vector bundles on ${}_\gamma\mathcal{F}$ equipped with left B-module structure. We prove that when F is a number field, $K_{2n+1}({}_\gamma\mathcal{F}, B)$ is a finitely generated abelian group and $K_{2n}({}_\gamma\mathcal{F}, B)$ is torsion and has no non-trivial divisible elements for all $n \geq 1$ (see Theorem 2.2). When F is a p-adic field, we prove that $K_{2n}({}_\gamma\mathcal{F}, B)_\ell$ is a finite group for all $n \geq 1$ (see Theorem 2.5).

Let ℓ be an odd rational prime and s a positive integer. In Sect. 3, we introduce mod-ℓ^s and profinite equivariant higher algebraic K-theory (which are not mentioned in the paper of Panin) and provide copious examples. In Sect. 4, we prove a general result that if $\mathcal{C}$ and $\mathcal{C}'$ are exact categories, then any isomorphism $f_* : K_n(C) \approx K_n(\mathcal{C}')$ induces isomorphisms $\hat{f}_* : K_n(\mathcal{C}, \mathrm{Z}/\ell^s) \approx K_n(\mathcal{C}', \mathrm{Z}/\ell^s)$ and $f_*^{\mathrm{pr}} : K_n^{\mathrm{pr}}(\mathcal{C}, \hat{\mathrm{Z}}_\ell) \approx K_n^{\mathrm{pr}}(\mathcal{C}', \hat{\mathrm{Z}}_\ell)$ (see Theorem 4.2). In 4.3, we apply this result to copious examples for various equivariant exact categories $\mathcal{C}$ and $\mathcal{C}'$. In particular, we deduce that if $\mathcal{C} = \mathcal{VB}_{\tilde{G}}({}_\gamma\mathcal{F}, B)$ and $\mathcal{C}' = \mathcal{P}(A_{\chi,\gamma} \otimes B)$, the category of finitely generated projective $A_{\chi,\gamma} \otimes B$-modules, then we have isomorphisms $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathrm{Z}}_\ell) \approx K_n^{\mathrm{pr}}((A_{\chi,\gamma} \otimes B), \hat{\mathrm{Z}}_\ell)$ which reduces computations of $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathrm{Z}}_\ell)$ to that of profinite higher K-theory of an arbitrary finite dimensional semi-simple algebras.

In Sect. 5, we are thus able to prove that if F is a number field then for all $n \geq 1$, $K_{2n}^{\mathrm{pr}}(({}_\gamma\mathcal{F}, \mathcal{B}), \hat{\mathrm{Z}}_\ell)$ is ℓ-complete and $\mathrm{div} K_{2n}^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathrm{Z}}_\ell) = 0$..

When F is a p-adic field, we have that for all $n \geq 1$, $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \mathrm{Z}_\ell) \approx K_n(({}_\gamma\mathcal{F}, B), \hat{\mathrm{Z}}_\ell)$ are ℓ-complete profinite groups, $\mathrm{div} K_{2n}^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{\mathrm{Z}}_\ell) = 0$.

Notes on Notation: For an additive abelian group A and a positive integer m, we write *A/m* for *A/mA*, and $A[m] = \{x \in A | mx = 0\}$. If ℓ is a rational prime, we denote by A_ℓ the ℓ-primary torsion subgroup of A, i.e., $A_\ell = \cup A[\ell^s] = \xrightarrow{\lim} A[\ell^s]$.

If $\mathcal{A}$ is an exact category in the sense of Quillen [14], we shall write $K_n(A) n \geq 0$ for Quillen's higher K-group $\pi_{n+1}(BQ\mathcal{A})$ (see [14]). If R is a ring with identity, and $\mathcal{P}(R)$ (resp. $\mathcal{M}(R)$) the category of finitely generated projective (resp. finitely generated) R-modules, we shall write $K_n(R)$, (resp. $G_n(R)$) for $K_n(\mathcal{P}(R))$ (resp. $K_n(\mathcal{M}(R))$)

Let F_{sep} be the separable closure of a field F, we shall write $\mathrm{Gal}(F_{\mathrm{sep}}/F)$ for the Galois group of F_{sep} over F.

1 Equivariant higher K-theory for schemes

In this section, we briefly review equivariant higher algebraic K-theory for schemes as defined by Thomason in [19], as well as review some relevant examples.

1.1 Generalities

1.1.1

Let G be an algebraic group over a field F and $Rep_F(G)$ the category of representations of G in the category $\mathcal{P}$(F) of finite dimensional vector spaces over F. We denote $K_0(Rep_F(G))$ by

$R_F(G)$ or $R(G,F)$ (or just $R(G)$ when the context is clear). Note that $R(G)$ is the free abelian group generated by the classes of irreducible representations and that $R(G)$ also has a ring structure induced by tensor product. Call $R(G)$ the representation ring.

Since $Rep_F(G)$is an exact category (see [10]), we denote $K_n(Rep_F(G))$ by $K_n(G, F)$, which is also equal to $G_n(G, F)$ (see [10]). So,$G_0(G, F) = R_F(G) = K_0(G, F)$ (see 1.1.3 below).

1.1.2

Let G be a group scheme over a scheme Y (we shall mostly be interested in $Y = \text{Spec}(F)$, F a field). A scheme X over Y is called a G-scheme if there is an action morphism $\theta : G \underset{Y}{\times} X \to X$ (see [19] or [11]).

A G-module M over X is a coherent O_X-module M together with an isomorphism of $G \underset{Y}{\times} X$-modules $\rho : \theta^*(M) \to p_2^*(M)$ where $p_2 : G \underset{Y}{\times} X \to X$ is the projection satisfying the cocycle identity on $G \underset{Y}{\times} G \underset{Y}{\times} X$:

$$p_{23}^*(\rho) \circ (\text{id}_G \times \theta)^* (\rho) = (m \times \text{id}_G)^* (\rho),$$

where $G \underset{Y}{\times} G \xrightarrow{m} G$ is the multiplication (see [11] or [19]).

1.1.3

Let $\mathcal{M}(G,X)$ denote the abelian category of G-modules over a G-scheme X. We write $G_n(G,X)$ for $K_n(\mathcal{M}(G, X))$. Note that when $X = \text{Spec}(F)$ we recover $G_n(G,F)$ in 1.1.1. If G is trivial, we write $G_n(G, X) = G_n(X)$.

Let $\mathcal{P}(G,X)$ be the full subcategory of $\mathcal{M}(G,X)$ consisting of locally free O_X-modules. We write $K_n(G, X)$ for$K_n(P(G,X))$.

1.1.4

We have the following generalization of 1.1.3 (see [11,13]):

Let A be a finite dimensional separable F-algebra, G an algebraic group over F and X a G-scheme. A G-A-module over a G-scheme X is a G-module M which is also a left $A \otimes_F O_X$-module such that $g(am) = ga \cdot gm \text{for} g \in G,\ m \in M$.

Let $\mathcal{M}(G, X, A)$ be the Abelian category whose objects are G-A-modules and whose morphisms are $A \otimes_F O_X$- and G-module morphisms. We write $G_n(G, X, A)$ for K_n $(M(G, X, A))$. Note that $\mathcal{M}(G, X, F) \approx \mathcal{M}(G,X)$ and so, $G_n(G, X, F) \approx G_n$ (G,X).

Let $\mathcal{P}(G, X, A)$ be the full subcategory of $\mathcal{M}(G, X, A)$ consisting of locally free $O_{A \otimes O_x}$-modules. Write $K_n(G, X, A)$ for $K_n(P(G, X, A))$. Hence $\mathcal{P}(G, X, F) \approx \mathcal{P}(G, X)$, $K_n(G, X, F) \approx K_n(G, X)$.

1.1.5

Let G be an affine algebraic group over F, X a G-scheme, $\mathcal{VB}_G(X)$ the category of G-equivariant vector bundles on X. If H is a closed subgroup of G, then we have an equivalence of categories

$$\text{Rep}_F\ (H) \underset{\text{res}}{\overset{\text{ind}}{\leftrightarrows}} \mathcal{VB}_G(G/H),$$

where 'ind' and 'res' are defined as follows:

- res: For any vector bundle $E \xrightarrow{p} G/H$, $p^{-1}(\bar{e}) \in \mathrm{Rep}_F(H)$ (where $\bar{e} = eH = H$) since the stabilizer of H in $G/H = \bar{e}$.
- ind: Let $(V, \alpha : H \to \mathrm{Aut}(V)) \in Rep_F(H)$. Then, one has a vector bundle $(G \times V)/H \to G/H$ where H acts on $(G \times V)/H$ by $h(g, v) = (gh^{-1}, \alpha(h)v)$. We denote $(G \times V)/H$ by $\tilde{V}$. So we get $K_n(Rep_F(H)) \approx K_n(\mathcal{VB}_G(G/H))$. We denote $K_n(\mathcal{VB}_G(G/H))$ by $K_n(G/H)$.

1.2 Higher K-theory of twisted flag varieties

In this subsection we briefly introduce twisted flag varieties and their ordinary higher algebraic K-theory. Details can be found in [13]. We say enough here to develop notations for later use.

1.2.1

Let $\tilde{G}$ be a semi-simple connected and simply connected, F-split algebraic group over a field F. Let $\tilde{T} \subset \tilde{G}$ be a maximal F-split torus of $\tilde{G}$, $\tilde{P} \subset \tilde{G}$ a parabolic subgroup of $\tilde{G}$ containing the torus $\tilde{T}$. The factor variety $F = \tilde{G}/\tilde{P}$ is smooth and projective (see [13,2]). Call $F = \tilde{G}/\tilde{P}$ a flag variety.

Let $N_{\tilde{G}}(\tilde{T})$ be the normalizer of $\tilde{T}$ in $\tilde{G}$, $W := N_{\tilde{G}}(\tilde{T})/\tilde{T}$ the Weyl group of $\tilde{G}$– a finite group. Let $W_{\tilde{P}} := \{w \in W | w\tilde{P}w^{-1} = \tilde{P}.\}$. Put $n(\mathcal{F}) = |W : W_{\tilde{P}}|$. Note that $R(\tilde{P})$ is a free $R(\tilde{G})$-module of rank $n(\tilde{F})$ (see [13]).

1.2.2

Let $\tilde{Z}$ be the center of $\tilde{G}$ and $\tilde{Z}^*$= Hom$(\tilde{Z}, G_m)$ the group of characters of $\tilde{Z}$. Note that $\tilde{Z}^*$ is a finite group.

Let $\chi \in \tilde{Z}^*$ and $Rep_F^{\chi}(\tilde{P})$ be the full subcategory of $Rep_F(\tilde{P})$ consisting of those $V \in Rep_F(\tilde{P})$ such that $\tilde{Z}$ acts on V by the character χ. The F-group scheme $\tilde{Z}$ acts on V by the character χ and hence on every $\tilde{V} = (\tilde{G} \times V)/\tilde{P} \in \mathcal{VB}_{\tilde{G}}(\mathcal{F})$ (see 1.1.5).

Let $\mathcal{VB}_{\tilde{G}}(\mathcal{F}, \chi)$ be the full subcategory of $\mathcal{VB}_{\tilde{G}}(\mathcal{F})$ consisting of those $\tilde{V}$ such that $\tilde{Z}$ acts on every fibre of $\tilde{V}$ by the character χ. Write $K_n(\mathcal{F}, \chi)$ for $K_n(\mathcal{VB}_{\tilde{G}}(\mathcal{F}, \chi))$ and $R^{\chi}(\tilde{P})$ for $K_0(Rep_F^{\chi}(\tilde{P}))$.

1.2.3

Let $\tilde{G}, \tilde{Z}, \tilde{T}, \tilde{P}$ be as in 1.2.1 and 1.2.2. Put $G = \tilde{G}/\tilde{Z}$, $P = \tilde{P}/\tilde{Z}$, $T = \tilde{T}/\tilde{Z}$ and $\mathcal{F} = \tilde{G}/\tilde{P} = G/P$. Put $\mathbf{g} = Gal(F_{\text{sep}}/F)$ where F_{sep} is the separable closure of F. Let $\gamma : \mathbf{g} \to G(F_{\text{sep}})$ be a l-cocycle (see [13]) and ${}_{\gamma}\mathcal{F}$ the twisted form of $\mathcal{F}$ corresponding to γ (see [11] or [13]). We write $K_n({}_{\gamma}\mathcal{F})$ for $K_n(\mathcal{VB}_G({}_{\gamma}\mathcal{F}))$.

Now, for $\chi \in \tilde{Z}^* = \mathrm{Hom}(\tilde{Z}, G_m)$, choose a non-trivial representation $V_{\chi} \in \mathrm{Rep}^{\chi}(\tilde{G})$. Put $A_{\chi} = \mathrm{End}_F(V_{\chi})$. Then A_{χ} is an F-algebra equipped with a G-action by F-algebra automorphism (see [13]). Using the 1-cocycle γ, one gets a new g-action on $A_{\chi} \otimes_F F_{\text{sep}}$ and hence a twisted form $A_{\chi,\gamma}$of the algebra A_{χ}(see [13]).

Lemma 1.2.4 [13] *Let F be a field. Assume that char* $(F) = 0$ *or that char (F) is prime to the order of* $\tilde{Z}^*$. *Then* $A_{\chi,\gamma}$ *is a finite dimensional central simple F-algebra.*

1.2.5

Let B be a finite dimensional separable F-algebra, X a smooth projective variety equipped with the action of an affine algebraic group G over F, ${}_\gamma X$ the twisted form of X via a 1-cocycle γ. Let $VB_G\left({}_\gamma X, B\right)$ be the category of vector bundles on ${}_\gamma X$ equipped with left B-module structure.

We now quote the following result due to I. Panin.

Theorem 1.2.6 [13] *In the notations of 1.2.3, 1.2.4 and 1.2.5, there exists an isomorphism* $f_* : K_n\left({}_\gamma\mathcal{F}, B\right) \approx K_n\left(A_{\chi,\gamma} \otimes B\right)$ *for all* $n \geq 1$.

2 Some finiteness results for higher *K*-theory of twisted flag varieties

Before discussing profinite equivariant higher K-theory which is the main focus of this paper, we first prove some preliminary results on ordinary higher K-theory of twisted flag varieties.

2.1

Let $\tilde{G}$ be a semi-simple, simply connected and connected F-split algebraic group over a field F, $\tilde{P}$ a parabolic subgroup of $\tilde{G}$, $F := \tilde{G}/\tilde{P}$, γ the 1-cocycle $\gamma : Gal(F_{\rm sep}/F) \to \tilde{G}(F_{\rm sep})$, ${}_\gamma\mathcal{F}$ the twisted form of $\mathcal{F}$. Let B be a finite dimensional separable F-algebra. We write $K_n({}_\gamma\mathcal{F}, B)$ for K_n of the category $VB_G({}_\gamma\mathcal{F}, B)$ of vector bundles on ${}_\gamma\mathcal{F}$ equipped with left B-module structure. We prove the following result.

Theorem 2.2 *Let F be a number field. Then for all* $n \geq 1$,

(a) $K_{2n+1}\left({}_\gamma\mathcal{F}, B\right)$ *is a finitely generated Abelian group.*
(b) $K_{2n}\left({}_\gamma\mathcal{F}, B\right)$ *is a torsion group and has no non-trivial divisible elements.*

Remarks 2.3 It follows from [14] p. 136 that in the statement of Theorem 1.2.6, $A_{\chi,\gamma} \otimes B$ is a finite dimensional semi-simple F-algebra. Hence, in order to prove Theorem 2.2 above, it suffices to prove (a), (b) of 2.2 for $K_{2n+1}\left(\sum\right)$, $K_{2n}\left(\sum\right)$ where $\sum$ is an arbitrary finite-dimensional semi-simple F-algebra. This we do in 2.4 below.

Theorem 2.4 *Let* $\sum$ *be a finite-dimensional semi-simple algebra over a number field F. Then for all* $n \geq 1$

(a) $K_{2n+1}\left(\sum\right)$ *is a finitely generated Abelian group.*
(b) $K_{2n}\left(\sum\right)$ *is torsion and has no non-zero divisible elements.*

Proof (a) Let R be the ring of integers of F. It is well-known that any semi-simple F-algebra contains at least one maximal R-order (see [10,16] or [4]). So let Γbe a maximal order in $\sum$. From the localization sequence

$$\cdots \to \underset{\underline{p}}{\oplus} G_{2n+1}\left(\Gamma/\underline{p}\Gamma\right) \to G_{2n+1}(\Gamma) \to G_{2n+1}(\textstyle\sum) \to \underset{\underline{p}}{\oplus} G_{2n}\left(\Gamma/\underline{p}\Gamma\right) \to \cdots \quad \text{(I)}$$

(whose $\underline{p}$ ranges over all prime ideals of R) we have

$$G_{2n}\left(\Gamma/\underline{p}\Gamma\right) \approx K_{2n}\left(\left(\Gamma/\underline{p}\Gamma\right)/\text{rad}\left(\Gamma/\underline{p}\Gamma\right)\right)$$

where $(\Gamma/\underline{p}\Gamma)/\text{rad}(\Gamma/\underline{p}\Gamma)$ is a finite semi-simple ring which is a direct product of matrix algebras over finite fields. So, $G_{2n}(\Gamma/\underline{p}\Gamma) = 0$. Note that since Γ and $\sum$ are regular,

$K_n(\Gamma) \approx G_n(\Gamma)$ and $K_n(\sum) \approx G_n(\sum)\ \forall\ n \geq 0$. But $K_{2n+1}(\Gamma)$ is finitely generated (see [10, theorem 7.1.13]) or [7]). Hence $K_{2n+1}(\sum)$ is finitely generated as a homomorphic image of $G_{2m+1}(\Gamma)$ □

(b) Recall from the proof of (a) that $G_{2n}\left(\Gamma/\underline{p}\Gamma\right) = 0$. Hence Quillen's localization sequence yields.

$$0 \to G_{2n}(\Gamma) \to G_{2n}\left(\sum\right) \to \underset{\underline{p}}{\oplus} G_{2n-1}\left(\Gamma/\underline{p}\Gamma\right) \to SK_{2n-1}(\Gamma) \to 0. \qquad \text{(II)}$$

Also, recall that since Γ, $\sum$ are regular, $K_n(\Gamma) \approx G_n(\Gamma)$ and $K_n(\sum) \approx G_n(\sum)\ \forall\ n \geq 0$. But $G_{2n}(\Gamma) \approx K_{2n}(\Gamma)$ is a finite group for all $n \geq 1$ (see [10] theorem 7.1.12 or [6]). Also, $\oplus G_{2n+1}\left(\Gamma/p\Gamma\right)$ is a torsion group as a direct sum of finite groups, see [10, 7.1.12]. Hence it follows from the diagram (II) above that $G_{2n}\left(\sum\right) \approx K_{2n}\left(\sum\right)$ is a torsion group.

Also from the sequence (II), $\oplus G_{2n-1}\left(\Gamma/p\Gamma\right)$, as a direct sum of finite groups has no non-trivial divisible elements. So any divisible element in $K_{2n}\left(\sum\right)$ must come from $G_{2n}(\Gamma) \approx K_{2n}(\Gamma)$. But $K_{2n}(\Gamma)$ is a finite group (see [9]) and also has no non-trivial divisible elements. Hence $G_{2n}\left(\sum\right)$ has no non-trivial divisible elements.

We now turn our attention to the local situation and prove the following:

Theorem 2.5 *Let F be a p-adic field, ℓ an odd rational prime such that $\ell \neq p$. Then for all $n \geq 1$ and any finite dimensional separable algebra B, $K_n\left({}_{\gamma}\mathcal{F}, \mathcal{B}\right)_{\ell}$ is a finite group.*

Remark 2.6 In view of Theorem 1.2.6, it suffices to show that $K_n\left(A_{\chi\gamma} \otimes B\right)_{\ell}$ is finite for all $n \geq 1$. To do this, it suffices to show that if $\sum$ is any finite dimensional semi-simple F-algebra, then $K_n\left(\sum\right)_{\ell}$ is a finite group. We state this formally as 2.7.

Theorem 2.7 *Let F be a p-adic field and ℓ an odd rational prime such that $\ell \neq p$. Suppose that $\sum$ is a finite dimensional semi-simple F-algebra. Then for all $n \geq 1$, $K_n\left(\sum\right)_{\ell}$ is a finite group.*

Proof Since $\sum$ is Morita equivalent to a finite product of matrix algebras over division algebras, it suffices to consider the case where $\sum = \mathrm{D}$ is a central division algebra over F. (Note that this might involve replacing this original F by finite extension field)

Now, D has at least one maximal order Γ, say (see [4]). Let $\underline{m}$ be the unique maximal ideal of Γ. Then, from the localization sequence

$$\cdots \to K_n\left(\Gamma/\underline{m}, \mathrm{Z}/\ell^s\right) \to K_n(\Gamma, \mathrm{Z}/\ell^s) \to K_n(D, \mathrm{Z}/\ell^s) \to K_{n-1}\left(\Gamma/\underline{m}, \mathrm{Z}/\ell^s\right) \to \cdots \qquad \text{(III)}$$

we know that $K_n(\Gamma, \mathrm{Z}/\ell^s) \approx K_n\left(\Gamma/\underline{m}, \mathrm{Z}/\ell^s\right)\ \forall\ n \geq 1$. (See [18, corollary 2 to theorem 2]). Note that we have been using properties of mod-ℓ^s K-theory defined in Sect. 3.1).

Now, the groups $K_n(\Gamma/\underline{m}, \mathrm{Z}/\ell^s)$, $n \geq 1$ are finite groups with orders uniformly bounded as functions of s (see [18]). Hence, so are the groups $K_n(D, \mathrm{Z}/\ell^s)$ and $K_n(\Gamma, \mathrm{Z}/\ell^s)$ (from the exact sequence (III)). Also from 3.1.2, we have an exact sequence

$$0 \to K_{n+1}(D)/\ell^s \to K_{n+1}(D, \mathrm{Z}/\ell^s) \to K_n(D)[\ell^s] \to 0 \qquad \text{(IV)}$$

where $K_{n+1}(D, \mathrm{Z}/\ell^s)$ is a finite group having orders uniformly bounded as functions of s (as seen above). So for fixed n, the groups $K_n(D)[\ell^s]$ are equal for s sufficiently large. But $K_n(D)_\ell = \cup_{n=1}^{\infty} K_n(D)\left[\ell^s\right]$. Hence $K_n(D)_\ell$ is finite. □

3 MOD-ℓ^s and profinite higher K-theory for schemes—definitions and relevant examples

In this section we briefly introduce mod-ℓ^s and profinite K-theory for exact categories with examples relevant to this paper. More details and examples can be found in [10, chapter 8] or [8].

3.1 Mod-ℓ^s K-theory of an exact category $\mathcal{C}$

3.1.1

Let $\mathcal{C}$ be an exact category, ℓ a rational prime, s a positive integer, $M_{\ell^s}^{n+1}$ the $(n+1)$-dimensional mod-ℓ^s Moore space, i.e., the space obtained from S^n by attaching an $(n+1)$-cell via a map of degree ℓ^s (see [3,12]).

If X is any H-space, write $\pi_{n+1}(X, \mathbb{Z}/\ell^s)$ for $[M_{\ell^s}^{n+1}, X]$, the set of homotopy classes of maps from $M_{\ell^s}^{n+1}$ to X. If $\mathcal{C}$ is an exact category and $X = BQ\mathcal{C}$, we write

$K_n(\mathcal{C}, \mathbb{Z}/\ell^s)$ for $\pi_{n+1}(BQ\mathcal{C}, \mathbb{Z}/\ell^s)$ for $n \geq 1$ and $K_0(\mathcal{C}, \mathbb{Z}/\ell^s)$ for $K_0(\mathcal{C}) \otimes \mathbb{Z}/\ell^s$. Call $K_n(\mathcal{C}, \mathbb{Z}/\ell^s)$ mod-ℓ^s K-theory of $\mathcal{C}$.

3.1.2

Note from [10, 8.1.12] or [8] that the exact sequence

$$\cdots K_n(\mathcal{C}) \xrightarrow{\ell^s} K_n(\mathcal{C}) \xrightarrow{\rho} K_n(\mathcal{C}, \mathbb{Z}/\ell^s) \xrightarrow{\beta} K_{n-1}(\mathcal{C}) \to K_{n-1}(\mathcal{C}) \to \cdots$$

induces a short exact sequence for all $n \geq 2$.

$$0 \to K_n(\mathcal{C})/\ell^s \to K_n(\mathcal{C}, \mathbb{Z}/\ell^s) \to K_{n-1}(\mathcal{C})\left[\ell^s\right] \to 0.$$

Examples 3.1.3 (i) If A is a ring with identity, and $\mathcal{C} = \mathcal{P}(A)$ the category of finitely generated projective A-modules, write $K_n(A, \mathbb{Z}/\ell^s)$ for $K_n(\mathcal{P}(A), \mathbb{Z}/\ell^s)$. Note that $K_n(A, \mathbb{Z}/\ell^s)$ is also $\pi_n((BGL(A)^+, \mathbb{Z}/\ell^s)$.

(ii) If Y is a scheme and $\mathcal{C} = \mathcal{P}(Y)$, the category of locally free sheaves of O_Y-modules, write $K_n(Y, \mathbb{Z}/\ell^s)$ for $K_n(\mathcal{P}(Y), \mathbb{Z}/\ell^s)$. Note that for $Y = \text{Spec}(A)$, A commutative, we recover $K_n(A, \mathbb{Z}/\ell^s)$.

(iii) Let A be a Noetherian ring and $\mathcal{M}(A)$ the category of finitely generated A-modules. We write

$$G_n(A, \mathbb{Z}/\ell^s) \quad \text{for } K_n(\mathcal{M}(A), \mathbb{Z}/\ell^s).$$

(iv) If Y is a Noetherian scheme, $\mathcal{C} = \mathcal{M}(Y)$ the category of coherent sheaves of O_Y-modules, we write

$$G_n(Y, \mathbb{Z}/\ell^s) \quad \text{for } G_n(\mathcal{M}(Y), \mathbb{Z}/\ell^s).$$

(v) Let G be an algebraic group over a field F, X a G-scheme and $\mathcal{C} = \mathcal{M}(G, X)$ as defined in 1.1.3, we write

$$G_n((G, X), \mathbb{Z}/\ell^s) \quad \text{for } K_n(\mathcal{M}(G, X), \mathbb{Z}/\ell^s).$$

(vi) If $\mathcal{C} = \mathcal{P}(G,X)$ as defined in 1.1.3, we write

$$K_n((G, X), \mathbb{Z}/\ell^s) \quad \text{for } K_n(\mathcal{P}(G, X), \mathbb{Z}/\ell^s).$$

(vii) If $\mathcal{C} = \mathcal{VB}_G\left({}_\gamma X, B\right)$ as in 1.2.5, we write

$$K_n(({}_\gamma X, B), \mathrm{Z}/\ell^s) \quad \text{for } K_n(\mathcal{VB}_G({}_\gamma X, B); \mathrm{Z}/\ell^s).$$

(viii) If $\mathcal{C} = \mathcal{M}(G, X, A)$ as defined in 1.1.4, we write

$$G_n((G, X, A), \mathrm{Z}/\ell^s) \quad \text{for } K_n(\mathcal{M}(G, X, A), \mathrm{Z}/\ell^s).$$

(ix) If $\mathcal{C} = \mathcal{P}((G, X, A)$ as in 1.1.4, we write

$$K_n((G, X, A), \mathrm{Z}/\ell^s) \quad \text{for } K_n(\mathcal{P}(G, X, A); \mathrm{Z}/\ell^s).$$

3.2 Profinite higher K-theory of an exact categories $\mathcal{C}$

3.2.1

Let $\mathcal{C}$ be an exact category, ℓ a rational prime, s a positive integer and n a non-negative integer. Put $M_\infty^{n+1} = \overset{\lim}{\longrightarrow} \; M_{\ell^s}^{n+1}$. We define the profinite higher K-theory of $\mathcal{C}$ by $K_n^{pr}(\mathcal{C}, \hat{\mathrm{Z}}_\ell) := [M_{l\infty}^{n+1}, BQ\mathcal{C}]$. We also write $K_n(\mathcal{C}, \hat{\mathrm{Z}}_\ell)$ for $\underset{s}{\overset{\lim}{\longleftarrow}} K_n(C, \mathrm{Z}/\ell^s)$. Note that for all $n \geq 1$, we have an exact sequence

$$0 \to \underset{s}{\overset{\lim}{\longleftarrow}} K_{n+1}\left(\mathcal{C}, Z/\ell^s\right) \to K_n^{pr}\left(\mathcal{C}, \hat{\mathrm{Z}}_\ell\right) \to K_n\left(\mathcal{C}, \hat{\mathrm{Z}}_\ell\right) \to 0.$$

For more information see [10] or [8].

Examples 3.2.2 (i) If $\mathcal{C} = \mathcal{P}(A)$ as in 3.1.3(i), we write $K_n^{\mathrm{pr}}(A, \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{P}(A), \hat{\mathrm{Z}}_\ell)$ and $K_n(A, \hat{\mathrm{Z}}_\ell)$ for $K_n(\mathcal{P}(A), \hat{\mathrm{Z}}_\ell)$.

(ii) If $\mathcal{C} = \mathcal{P}(Y)$ as in 3.1.3(ii), we write $K_n^{\mathrm{pr}}(Y, \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{P}(Y), \hat{\mathrm{Z}}_\ell)$ and $K_n(Y, \hat{\mathrm{Z}}_\ell)$ for $K_n(\mathcal{P}(Y), \hat{\mathrm{Z}}_\ell)$.

(iii) If $\mathcal{C} = \mathcal{M}(A)$ as in 3.1.3(iii) we write $G_n^{\mathrm{pr}}(A, \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{M}(A), \hat{\mathrm{Z}}_\ell)$ and $G_n(A, \hat{\mathrm{Z}}_\ell)$ for $K_n(\mathcal{M}(A), \hat{\mathrm{Z}}_\ell)$.

(iv) If $\mathcal{C} = \mathcal{M}(Y)$ as in 3.1.3(iv) we write $G_n^{\mathrm{pr}}(Y, \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{M}(Y), \hat{\mathrm{Z}}_\ell)$ and $G_n(Y, \hat{\mathrm{Z}}_\ell)$ for $K_n(\mathcal{M}(Y), \hat{\mathrm{Z}}_\ell)$.

(v) If $\mathcal{C} = \mathcal{M}(G, X)$ as in 3.1.3(v), we write $G_n^{\mathrm{pr}}((G, X), \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{M}(G, X), \hat{\mathrm{Z}}_\ell)$ and $G_n((G, X), \hat{\mathrm{Z}}_\ell)$ for $K_n(\mathcal{M}(G, X), \hat{\mathrm{Z}}_\ell)$.

(vi) If $\mathcal{C} = \mathcal{P}(G, X)$ as in 3.1.3(vi), we write $K_n^{\mathrm{pr}}((G, X), \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{P}(G, X), \hat{\mathrm{Z}}_\ell)$ and $K_n((G, X), \hat{\mathrm{Z}}_\ell)$ for $K_n(\mathcal{P}(G, X), \hat{\mathrm{Z}}_\ell)$.

(vii) If $\mathcal{C} = \mathcal{VB}_G({}_\gamma X, B)$ as in 3.1.3(vii), we write $K_n^{\mathrm{pr}}(({}_\gamma X, B), \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{V}\mathcal{B}_G({}_\gamma X, B), \hat{\mathrm{Z}}_\ell)$.

(viii) If $\mathcal{C} = \mathcal{M}(G, X, A)$ as in 3.1.3(viii), we write $G_n^{\mathrm{pr}}((G, X, A), \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{M}(G, X, A), \hat{\mathrm{Z}}_\ell)$ and $G_n((G, X, A), \hat{\mathrm{Z}}_\ell)$ for $K_n(\mathcal{M}(G, X, A), \hat{\mathrm{Z}}_\ell)$.

(ix) If $\mathcal{C} = \mathcal{P}(G, X, A)$ as in 3.1.3(ix), we write $K_n^{\mathrm{pr}}((G, X, A), \hat{\mathrm{Z}}_\ell)$ for $K_n^{\mathrm{pr}}(\mathcal{P}(G, X, A), \hat{\mathrm{Z}}_\ell)$ and $K_n((G, X, A), \hat{\mathrm{Z}}_\ell)$ for $K_n(P(G, X, A), \hat{\mathrm{Z}}_\ell)$.

4 A general result connecting mod-ℓ^s (resp.profinite) higher *K*-theory of two exact categories

4.1

Let $\mathcal{C}, \mathcal{C}'$ be two exact categories. Suppose that there exists an isomorphism $f_* : K_n(C) \approx K_n(\mathcal{C}')$ for $n \geq 1$ induced by an exact functor f : C $\rightarrow$ C'.We prove, in this section, a result connecting $K_n(\mathcal{C}, Z/\ell^s)$ and $K_n(\mathcal{C}', Z/\ell^s)$ as well as that connecting $K_n^{\mathrm{pr}}(\mathcal{C}, \hat{Z}_l)$ and $K_n^{\mathrm{pr}}(\mathcal{C}', \hat{Z}_\ell)$.

More precisely, we prove the following:

Theorem 4.2 *Let $\mathcal{C}, \mathcal{C}'$ be exact categories such that there exists an isomorphism f_* : $K_n(\mathcal{C}) \approx K_n(\mathcal{C}')$ for each $n \geq 1$ induced by an exact functor $f : C \rightarrow C'$. Let ℓ be an odd rational prime and s a positive integer. Then the induced homomorphisms*

$$\hat{f}_* : K_n\left(\mathcal{C}, Z/\ell^s\right) \rightarrow K_n\left(\mathcal{C}', Z/\ell^s\right)$$

and

$$f_*^{\mathrm{pr}} : K_n^{\mathrm{pr}}\left(C, \hat{Z}_\ell\right) \rightarrow K_n^{\mathrm{pr}}\left(\mathcal{C}', \hat{Z}_\ell\right)$$

are also isomorphisms.

Proof Consider the following commutative diagram (I) where the rows are exact and the vertical arrows are induced from f_*.

$$\begin{array}{ccccccccc} 0 \rightarrow & K_n(\mathcal{C})/\ell^s & \xrightarrow{\delta} & K_n\left(\mathcal{C}, Z/\ell^s\right) & \xrightarrow{\eta} & K_{n-1}(\mathcal{C})\left[\ell^s\right] & \rightarrow 0 \\ & \downarrow \bar{f}_* & & \downarrow \hat{f}_* & & \downarrow f'_* & \\ 0 \rightarrow & K_n(\mathcal{C}')/\ell^s & \xrightarrow{\delta'} & K_n\left(\mathcal{C}', Z/\ell^s\right) & \xrightarrow{\eta'} & K_{n-1}(\mathcal{C}')\left[\ell^s\right] & \rightarrow 0 \end{array} \qquad \text{(I)}$$

Put $A = K_n(\mathcal{C})$, $B = K_n(\mathcal{C}')$. By hypothesis, $f_* : A \rightarrow B$ is a homomorphism and so, the induced vertical maps $\bar{f}_*$, $\hat{f}_*$ and f'_* are homomorphism. We show that they are bijective. □

By assuming that $f_* : A \rightarrow B$ is bijective, we show that the induced map $f_* : A/\ell^s A \rightarrow B/\ell^s B$ is bijective. So let $\bar{b} = b + \ell^s B \in B/\ell^s B$. Since f_* is surjective then $B = f_*(A)$ and there exists $a \in A$ such that $b = f_*(a)$. Then $\bar{b} = f_*(a) + \ell^s f_*(a) \in f_*(a + \ell^s A)$. So, $\bar{f}_*$ is surjective.

Now, let $a + \ell^s A =: \bar{a} \in \ker \bar{f}_*$. Then $f_*(a) + \ell^s B = \ell^s B$ i.e., $f_*(a) \in \ell^s f_*(A)$. So, $f_*(a) = \ell^s f(a')$ for some $a' \in A$. So, $f_*(a - \ell^s a') = 0$. Since f_* is injective, we have that $a = \ell^s a'$, i.e., $a \in \ell^s A$. So, $\bar{a} = 0$. So $\bar{f}_*$ is also injective. Hence $\bar{f}_*$ is bijective.

Now, $f_* : A \rightarrow B$ induces a map $f'_* : A[\ell^s] \rightarrow B[\ell^s]$. Indeed, if $a \in A[\ell^s]$, then $\ell^s a = 0$ in A, and so $\ell^s f_*(a) = 0$ in B. Hence, $f_*(a) \in B[\ell^s]$.

Now, let $b \in B[\ell^s]$. Then $\ell^s b = 0$. Since f_* is surjective, there exists $a \in A$ such that $b = f_*(a)$. So, $\ell^s b = \ell^s f_*(a) = f_*(\ell^s a) = 0$. But f_* is also injective and so, $\ell^s a = 0$, i.e., $a \in A[\ell^s]$. So f'_* is surjective. Now let $a \in \ker f'_*$, then $f'_*(a) = 0$. But $f'_*(a) = 0 \Rightarrow f_*(a) = 0$ since $f'_* = f_*|A[\ell^s]$. Since f_* is injective, $a = 0$. So f'_* is injective; so f'_* is bijective.

By applying the five lemma to diagram I, we have that $\hat{f}_* : K_n(\mathcal{C}, Z/\ell^s) \rightarrow K_n(\mathcal{C}', Z/\ell^s)$ is bijective. This proves the first part.

Now consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \varprojlim_s^1 K_{n+1}(\mathcal{C}, Z/\ell^s) & \xrightarrow{\delta} & K_n^{\mathrm{pr}}\left(\mathcal{C}, \hat{Z}_\ell\right) & \xrightarrow{\eta} & K_n\left(\mathcal{C}, , \hat{Z}_\ell\right) & \to 0 \\ & \downarrow \hat{f}'_* & & \downarrow f_*^{pr} & & \downarrow \hat{f}''_* & \\ 0 \to & \varprojlim_s^1 K_n\left(\mathcal{C}', Z/\ell^s\right) & \xrightarrow{\delta'} & K_n^{pr}\left(\mathcal{C}', \hat{Z}_\ell\right) & \xrightarrow{\eta'} & K_n\left(\mathcal{C}', \hat{Z}_\ell\right) & \to 0 \end{array} \tag{II}$$

where $\hat{f}'_*$ and $\hat{f}''_*$are induced by $\hat{f}_*$ in diagram (I).

Now, since from the first part, the homomorphism $\hat{f}_* : K_n(\mathcal{C}, Z/\ell^s) \to K_n(\mathcal{C}', Z/\ell^s)$ is bijective, it follows that the homomorphism

$$\hat{f}'_* : \varprojlim_s^1 K_{n+1}\left(\mathcal{C}, Z/\ell^s\right) \to \varprojlim_s^1 K_{n+1}\left(\mathcal{C}', Z/\ell^s\right)_s$$

is bijective and that

$$K_n\left(\mathcal{C}', Z_\ell\right) := \varprojlim_s K_n\left(\mathcal{C}', Z/\ell^s\right) \to \varprojlim_s K_n\left(\mathcal{C}', Z/\ell^s\right) = K_n\left(\mathcal{C}', \hat{Z}_\ell\right)$$

is bijective. Hence by applying the five lemma to diagram II, we have that that the middle homomorphism f_*^{pr} is an isomorphism. This proves the second part.

4.3 Remarks and examples

1. Let $\tilde{G}, \tilde{T}, \tilde{P}$ be as in 1.2.1, $\mathcal{F} = \tilde{G}/\tilde{P}$ a flag variety, $g = \mathrm{Gal}(F_{\mathrm{sep}}/.F)$ where F_{sep} is the separable closure of F. Let $\gamma : g \to G(F_{\mathrm{sep}})$ the 1-cocycle, ${}_\gamma\mathcal{F}$ the twisted form of $\mathcal{F}$ corresponding to γ, (see [11] or [13]), B a finite-dimensional separable algebra and $\mathcal{VB}_G({}_\gamma\mathcal{F}, B)$ the exact category of vector bundles on ${}_\gamma\mathcal{F}$ equipped with left B-module structure. Then by Theorem 1.2.6, $K_n(\mathcal{VB}_G({}_\gamma\mathcal{F}, B)) := K_n({}_\gamma\mathcal{F}, B) \approx K_n(A_{\chi,\gamma} \otimes B) := K_n(\mathcal{P}(A_{\chi,\gamma} \otimes B))$ where $A_{\chi,\gamma} \otimes B$ is a finite dimensional semi-simple algebra. Put $\mathcal{C} = \mathcal{VB}_G({}_\gamma\mathcal{F}, B)$ and $\mathcal{C}' = \mathcal{P}(A_{\chi,\gamma} \otimes B)$, it follows from 4.2 that f_* induces isomorphism $\hat{f}_* : K_n(\mathcal{C}, Z/\ell^s) \approx K_n(\mathcal{C}', Z/\ell^s)$ and isomorphism $f_*^{\mathrm{pr}} : K_n^{\mathrm{pr}}(\mathcal{C}, \hat{Z}_\ell). \approx K_n^{\mathrm{pr}}(\mathcal{C}', \hat{Z}_\ell)$. If we put $A_{\chi,\gamma} \otimes B = \sum$, then in the notation of 3.1.3(i)–(vii) and 3.2.2(i)–(vii), we have $\hat{f}_* : K_n(({}_\gamma\mathcal{F}, B); Z/\ell^s) \approx K_n(\sum, Z/\ell^s)$ and $f_*^{\mathrm{pr}} : K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B); Z_\ell) \approx K_n^{\mathrm{pr}}(\sum, \hat{Z}_\ell)$.
2. Let H be a split solvable group, $T \subset H$ a split maximal torus. Suppose that $\mathcal{M}(H,X)$ (resp. $\mathcal{M}(T,X)$) is the Abelian category of H-modules (resp. T-modules over a G-scheme X (also a T-scheme by restriction)). Then by [11], the homomorphism $f_* : G_n(H, X) \to G_n(T, X)$ induced by the 'restriction' functor $\mathcal{M}(H, X) \xrightarrow{f} \mathcal{M}(T, X)$ is an isomorphism. Hence by 4.2, the induced homomorphism $f_* : G_n((H, X), Z/\ell^s) \to ((T, X), Z/\ell^s)$ and $f_*^{\mathrm{pr}} : G_n^{\mathrm{pr}}((H, X); \hat{Z}_\ell) \to G_n^{\mathrm{pr}}((T, X); \hat{Z}_\ell)$. are isomorphisms.
3. Let G be an algebraic group over a field F, H a closed subgroup of G such that $G/H \approx A_F^1$ and X a G-scheme. Then by [11] the restriction functor: $\mathcal{M}(G, X) \to \mathcal{M}(H, X)$ induces isomorphism $G_n(G, X) \approx G_n(H, X)$. Hence by 4.2, we have isomorphism

   $$G_n\left((G, X), Z/\ell^s\right) \approx G_n\left((H, X), Z/\ell^s\right)$$

   and

   $$G_n^{pr}\left((G, X), Z_\ell\right) \approx G_n^{pr}\left((H, X), \hat{Z}_\ell\right).$$

4. Let G be an algebraic group over a field F and X a quasi-projective smooth G-scheme. Let $\mathcal{M}(G, X, A)$ be the Abelian category whose objects are G-A-modules and whose morphisms are $A \otimes_F O_X$- and G-module morphisms. Let $\mathcal{P}(G, X, A)$ be the full subcategory of $\mathcal{M}(G, X, A)$ consisting of locally free $O_{A \bigotimes_o X}$-modules. Then by [11], the inclusion functor $f : \mathcal{P}(G, X, A) \to \mathcal{M}(G, X, A)$ induces isomorphism $K_n(G, X, A) \approx G_n(G, X, A)$. Hence by 4.2, we have isomorphism.

$$\hat{f}_* : K_n\left((G, X, A), \mathrm{Z}/\ell^s\right) \approx G_n\left((G, X, A), \mathrm{Z}/\ell^s\right)$$

and

$$f_*^{pr} : K_n^{pr}\left((G, X, A), \hat{\mathrm{Z}}_\ell\right) \approx G_n^{pr}\left((G, X, A), \hat{\mathrm{Z}}_\ell\right).$$

5. Let U be a split unipotent group over F, X a U-scheme. Then by [11], the restriction functor $f : \mathcal{M}(U, X) \to \mathcal{M}(X)$ induces an isomorphisms $K_n(U, X, A) \approx G_n(U, X, A)$. Hence by 4.2, we have isomorphism

$$\hat{f}_* : G_n\left((U, X), \mathrm{Z}/\ell^s\right) \approx G_n\left(X, \mathrm{Z}/\ell^s\right)$$

and

$$f_*^{pr} : G_n^{pr}\left((U, X), \hat{\mathrm{Z}}_\ell\right) \approx G_n^{pr}\left(X, \hat{\mathrm{Z}}_\ell\right).$$

5 Some computations on profinite higher *K*-theory of twisted flag varieties

5.1

In this section, we obtain some ℓ-completeness and other results for twisted flag varieties over number fields and p-adic fields. Recall that if $\ell$is a rational prime, an Abelian group H is said to be ℓ-complete if $H = \underset{\underset{s}{\longleftarrow}}{\lim} H/\ell^s H$.

Theorem 5.2 *Let F be a number field. Then, in the notations of* 1.1.3, 1.1.4, 1.1.5, *and* 4.3(1), *we have that for* $n \geq 1$,

(a) $K_{2n}^{pr}\left(\left({}_\gamma F, B\right); \hat{\mathrm{Z}}_\ell\right)$ *is an ℓ-complete Abelian group.*
(b) *div* $K_{2n}^{pr}\left(\left({}_\gamma F, B\right); \hat{\mathrm{Z}}_\ell\right) = 0$.

Remarks 5.3 From 4.2 and 4.3(1), we have that for all $n \geq 1$, $K_n^{pr}(({}_\gamma\mathcal{F}, B); \hat{\mathrm{Z}}_\ell) \approx K_n^{\mathrm{pr}}(A_{\chi,\gamma} \otimes B; \hat{\mathrm{Z}}_\ell)$ where $A_{\chi,\gamma} \otimes B$ is a finite-dimensional semi-simple F-algebra. So, to prove 5.2, it suffices to prove the following:

Theorem 5.4 *Let F be a number field. Then for all* $n \geq 1$,

(a) $K_{2n}^{\mathrm{pr}}\left(A_{\chi,\gamma} \otimes B; \hat{\mathrm{Z}}_\ell\right)$ *is an ℓ-complete Abelian group;*
(b) *div* $K_{2n}^{\mathrm{pr}}\left(A_{\chi,\gamma} \otimes B; \hat{\mathrm{Z}}_\ell\right) = 0$.

Proof Put $\sum = A_{\chi,\gamma} \otimes B$. Since $\sum$ is a finite dimensional semi-simple F-algebra then by Theorem 2.4, $K_{2n+1}(\sum)$ is a finitely generated Abelian group.

Now it is proved in [10, lemma 2.8] or [8], that for all $m \geq 2$ and any exact category $\mathcal{C}$,

$$\underset{\underset{s}{\longleftarrow}}{\lim} K_m^{\mathrm{pr}}\left(C, \hat{\mathrm{Z}}_\ell\right)/\ell^s \approx K_m\left(C, \hat{\mathrm{Z}}_\ell\right).$$

Hence for any $m \geq 2$

$$\varprojlim_s K_m^{\mathrm{pr}}\left(\sum, \hat{\mathrm{Z}}_\ell\right)/\ell^s \approx K_m\left(\sum, \hat{\mathrm{Z}}_\ell\right). \tag{III}$$

Also, for any $m \geq 2$ and any exact category $\mathcal{C}$, we have from [10, lemma 8.2.1] an exact sequence

$$0 \to \varprojlim_s{}^1 K_{m+1}\left(\mathcal{C}, \mathrm{Z}/\ell^s\right) \to K_m^{pr}\left(\mathcal{C}, \hat{\mathrm{Z}}_\ell\right) \to K_m\left(\mathcal{C}, \hat{\mathrm{Z}}_\ell\right) \to 0.$$

Hence we have an exact sequence (for $m \geq 2$)

$$0 \to \varprojlim_s{}^1 K_{m+1}\left(\sum, \mathrm{Z}/\ell^s\right) \to K_m^{\mathrm{pr}}\left(\sum, \hat{\mathrm{Z}}_\ell\right) \to K_m\left(\sum, \hat{\mathrm{Z}}_\ell\right) \to 0. \tag{IV}$$

Since $K_{2n+1}(\sum)$ is finitely generated for $n \geq 2$ then $K_{2n+1}(\sum, \mathrm{Z}/\ell^s)$ is a finite group and so, $\varprojlim_s{}^1 K_{2n+1}(\sum, Z/\ell^s) = 0$. Hence from (IV),

$$K_{2n}^{\mathrm{pr}}\left(\sum, \hat{\mathrm{Z}}_\ell\right) \approx K_{2n}\left(\sum, \hat{\mathrm{Z}}_\ell\right). \tag{V}$$

Also from (III),

$$\varprojlim_s K_{2n}^{\mathrm{pr}}\left(\sum, \hat{\mathrm{Z}}_\ell\right)/\ell^s \approx K_{2n}\left(\sum, \hat{\mathrm{Z}}_\ell\right). \tag{VI}$$

From (V) and (VI), we now have

$$\varprojlim_s K_{2n}^{\mathrm{pr}}\left(\sum, \hat{\mathrm{Z}}_\ell\right)/\ell^s \approx K_{2n}^{\mathrm{pr}}\left(\sum, \hat{\mathrm{Z}}_\ell\right).$$

So, $K_{2n}^{\mathrm{pr}}(\sum, \hat{\mathrm{Z}}_\ell)$ is ℓ-complete.

(b) From [10, theorem 8.2.2(ii)] or [8], we have that for all $m \geq 2$ and any exact category $\mathcal{C}$,

$$\varprojlim_s{}^1 K_{m+1}\left(C, \mathrm{Z}/\ell^s\right) = \mathrm{div} K_m^{\mathrm{pr}}\left(C, \hat{\mathrm{Z}}_\ell\right).$$

Hence for all $m \geq 2$

$$\varprojlim_s{}^1 K_{m+1}\left(\sum, \mathrm{Z}/\ell^s\right) = \mathrm{div} K_m^{\mathrm{pr}}\left(\sum, \hat{\mathrm{Z}}_\ell\right).$$

If $m = 2n$, then $K_{2n+1}(\sum)$ is finitely generated and so, $K_{2n+1}(\sum, \mathrm{Z}/\ell^s)$ is a finite group. Hence $\varprojlim_s{}^1 K_{2n+1}(\sum, \mathrm{Z}/\ell^s) = 0$. Hence div $K_{2n+1}^{\mathrm{pr}}(\sum, \hat{\mathrm{Z}}_\ell) = 0$. □

Theorem 5.5 *Let p be a rational prime, F a p-adic field, $\tilde{G}$ a semi-simple connected and simply connected split algebraic group over F, $\tilde{P}$ a parabolic subgroup of $\tilde{G}$, γ a 1-cocycle Gal $(F_{\mathrm{sep}}/F) \to \tilde{G}(F_{\mathrm{sep}})$, ${}_\gamma\mathcal{F}$ the γ-twisted form of F,B a finite dimensional separable F-algebra, ℓ a rational prime such that $\ell \neq p$. Then for all $n \geq 2$*

(a) $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B)\hat{Z}_\ell)$ *is an ℓ-complete profinite Abelian group;*
(b) $K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{Z}_\ell) \approx K_n(({}_\gamma F, B), \hat{Z}_\ell)$.
(c) $div K_n^{\mathrm{pr}}(({}_\gamma\mathcal{F}, B), \hat{Z}_\ell) = 0$ *for all $n \geq 2$.*

Remarks 5.6 In view of 4.2 and 4.3(1), it suffices to prove the following:

Theorem 5.7 *Let p be an odd rational prime, F a p-adic field. Let $\sum = A_{\chi,\gamma} \otimes B$. Then for all $n \geq 2$.*

(a) $K_n^{\text{pr}}(\sum; \hat{Z}_\ell)$ *is an ℓ-complete profinite Abelian group.*
(b) $K_n^{\text{pr}}(\sum; \hat{Z}_\ell) \approx K_n(\sum; \hat{Z}_\ell)$
(c) *div* $K_n^{\text{pr}}(\sum; \hat{Z}_\ell) = 0$.

Proof (a), (b). Since $K_n(A_{\chi,\gamma} \otimes_F B) \approx K_n({}_\gamma\mathcal{F}, B)$ and $A_{\chi,\gamma} \otimes_F B$is a semi-simple F-algebra $\sum$, say, and $\sum$ is Morita equivalent to a finite product of matrix algebras over division algebras, it suffices to prove (a), (b), (c) above by replacing $\sum$ with a central division algebra D over a p-adic field F. From the proof of Theorem 2.7, we saw already that $K_n(D, Z/\ell^s)$ is a finite group. Hence, in the exact sequence

$$0 \to \underset{s}{\overset{\lim^1}{\longleftarrow}} K_n\left(D, Z/\ell^s\right) \to K_n^{\text{pr}}\left(D, \hat{Z}_\ell\right) \to K_n\left(D, \hat{Z}_\ell\right) \to 0,$$

we have $\overset{\lim^1}{\longleftarrow} K_{n+1}(D, Z/\ell^s) = 0$. Hence

$$K_{n+1}^{\text{pr}}\left(D, \hat{Z}_\ell\right) \approx K_n\left(D, \hat{Z}_\ell\right) \tag{VII}$$

proving (b).

Now, for any exact category $\mathcal{C}$, we have $\overset{\lim}{\longleftarrow} (K_n^{\text{pr}}(\mathcal{C}, \hat{Z}_\ell)/\ell^s) \approx K_n(\mathcal{C}, \hat{Z}_\ell)$ for all $n \geq 2$ (see [10, lemma 8.2.2] or [7]). So, we have

$$\underset{s}{\overset{\lim}{\longleftarrow}} K_n^{\text{pr}}\left(D, Z\right)/\ell^s \approx K_n\left(D, \hat{Z}_\ell\right). \tag{VIII}$$

From (VII) and (VIII) we now have $\underset{s}{\overset{\lim}{\longleftarrow}} K_n^{\text{pr}}(D, \hat{Z}_\ell)/\ell^s \approx K_n^{\text{pr}}(D, \hat{Z}_\ell)$—proving (a). It is profinite because $K_n^{\text{pr}}(D, \hat{Z}_\ell) = \underset{s}{\overset{\lim}{\longleftarrow}} K_n(D, Z/\ell^s)$, where $K_n(D, Z/\ell^s)$ is a finite group.

(c) We saw in the proof of Theorem 2.7 that $K_n(D, Z/\ell^s)$ is a finite group for all $n \geq 2$. Hence $\underset{s}{\overset{\lim^1}{\longleftarrow}} K_n(D, Z/\ell^s) = 0$ for all $n \geq 2$. But by [10, theorem 8.2.2(ii)] or [8]

$$\overset{\lim^1}{\longleftarrow} K_{m+1}\left(D, Z/\ell^s\right) \approx \text{div } K_m^{\text{pr}}\left(D, \hat{Z}_\ell\right)$$

Hence $\text{div} K_n^{\text{pr}}(D, \hat{Z}_\ell) = 0$ as required for all $n \geq 1$. □

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