Some review of Professor Kuku's work

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Overview-(1)

- Professor Kuku's research area is Algebra in a broad sense. His research over the years have focussed on Commutative and Non-commutative Algebra /Arithmetic/Geometry through methods of K-theory, Cyclic homology, encompassing Algebra, Number theory, Representation theory, Algebraic topology, operator algebras and some Algebraic Geometry and Differential Geometry.

Overview-(2)

- Such usually non-commutative structures; include e.g. (1) Orders in algebras over number fields and p-adic fields; (2) Group-rings and representations of finite, discrete, profinite, algebraic and compact Lie groups); (3) C*-algebras, and Lie groups C*-algebras; (4) Hopf algebras and Quantum groups. Note that cyclic homology and K-theory of the latter two structures belongs to non-commutative geometry.

LF and NF functors with applications to the computation of Picard group of Algebraic Geometry

- His initial work contributed to the understanding of the LF and NF functors with applications to the computation of Picard group of Algebraic Geometry, see [3]. Moreover, he also contributed to the understanding of Whitehead groups of group-rings of finite group over the ring of integers in algebraic number fields and p-adic fields as well as Whitehead groups of orders in algebras over such fields. He proved several finiteness results in this direction (see [4],[5]).

[3]A.O. Kuku (1973): Some Algebraic K-theoretic applications of the LF
and NF functors. **Proceedings of the American Mathematical Society**,37 (2) 363-365.

[4]A.O. Kuku (1973): Whitehead group of orders in p-adic Semi-simple algebras. **Journal of Algebra** 25 (3) 415-418

[5] A.O. Kuku (1976): Some finiteness theorems in the K-theory of orders in p-adic algebras. **Journal of London Mathematical Society**, (13) 122-128.

Finiteness of K_n -(1)

- Later, with the definition of Higher Algebraic K-theory by Quillen and others, it became important for various applications to understand the structure of Higher K-theory of orders and group-rings, (that is, to study K_n (all n) of the category of finitely generated projective modules over grouprings and orders; K_n of the category of G-representations in the category of finitely generated modules over such rings as the ring of integers in number field, their localisations and completions, where G is a finite, profinite or compact group).

Finiteness of K_n -(2)

- More precisely, let R be the ring of integers in a number field F, A any R-order in a semi-simple F-algebra S, $\underline{p}$ a prime ideal of R, he has proved many striking results on the Higher K-groups $K_n(A)$, $G_n(A)$, as well as on Higher dimensional class groups $Cl_n(A)$. For example, he proved that for all $n \geq 1$, $K_n(A)$, $G_n(A)$ are finitely generated Abelian groups, and that rank $K_n(A)$ = rank G_n (A) = rank $K_n(S)$ for all $n \geq 2$; that SK_n (A), $SG_n(A)$, $SK_n(A\underline{p})$, $SG_n(\underline{p})$ and $Cl_n(A)$ are finite groups for all $n \geq 1$. See [10],[12],[14],[17],[21]. All the results above hold for A=RG, (G finite group) and he also proved that if G is a finite p-group, then $SK_n(RG)$, $Cl_n(RG)$ are finite p-groups for all $n \geq 1$. He also proved a striking characterisation of p-adic semisimple algebras S in terms of K-theory of maximal orders G in S, i.e. S is unramified over its centre if and only if $SK_{2n-1}(G)=0$ for all $n \geq 1$. (See [7]).

- [7] A.O. Kuku (1979): SGn of orders and group-rings. **Mathematisches Zeitschfifts,**, 291-295.
- [10] A.O. Kuku (1982): Higher algebraic K-theory of group-rings and orders in algebras over number fields. **Communications in Algebra.** 10(8) 905-916.
- [12] A.O. Kuku (1984): K-theory of group-rings of finite groups over maximal orders in division algebras. **Journal of Algebra**, 91(1) 19-31.

- [14] A.O. Kuku (1986): Kn, SKn of integral group-rings and orders. **Contemporary Mathematics**, 55, 333-338.
- [17] A.O. Kuku (1987): Some finiteness results in the higher algebraic K-theory of orders and group-rings, **Topology and its Applications 25,** 185-191.
- [21] A.O. Kuku (1999): Ranks of Kn and Gn of orders and group-rings of finite groups over integers in number fields **Journal of pure and Applied Algebra** 138 (1999), 39-44.

Higher K-theory of the category

- In [12], he obtained several important results on the Higher K-theory of the category of representations of a finite group G in the category of $\underline{P}(G)$ where G is a maximal order in a central division algebras over number fields and p-adic fields. These results translate into computations of $G_n(\Gamma G)$ as well as lead to showing via topological and representation theoretic techniques that a non-commutative analogue of a fundamental result of R.G. Swan at the zero-dimensional level does not hold.

equivariant Higher Algebraic K-theory

- Moreover, in collaboration with A. Dress, he was able to formulate an equivariant Higher Algebraic K-theory via the theory of Mackey functors and this equivariant theory has proved very useful in proving result on Higher K-theory of group-rings. More precisely, if G is any finite group, *C* an exact category, and T a G-set, they constructed higher algebraic K-functors , , as "Mackey functors" from the category of G-sets to the category of Abelian groups, (i.e. functors satisfying certain functorial properties, in particular, the categorical version of Mackey subgroup theorem in representation theory), in such a way as to identify with Kn(M(RH) = Gn(RH); with Kn(PR(RH))=Gn(R,H), and , with Kn(RH) for any subgroup H of G where for any ring R with identity, P(R) is the category of finitely generated projective R-modules and M(R) is the category of finitely generated R-modules, (R Noetherian), and P(RH) the category of RH-lattices, see [8],[9].

[8] A. Dress and A.O. Kuku (1981): The Cartan map for equivariant higher algebraic K-groups. **Communications in Algebra**, 9(1) 727-746.
[9] A.O Kuku and A. Dress (1982): A convenient setting for equivariant higher algebraic K-theory. **Lecture Notes in Mathematics. 966, Springer-Verlag; Berlin,** Heidelberg, New York, 59-68.

Generalization of these constructions

- He has since generalised these constructions to the cases where G is a profinite group (see [11]) and G a compact Lie group (see [24]). These constructions have also been useful in studying cohomology of groups. His book on 'Axiomatic theory of induced Representation of Finite Groups', is an exposition of the theory of Mackey functors in the context of representation of finite groups (see [48]). Through the above techniques he was able to prove the striking result that if k is a field of characteristic p, G a finite or profinite group, then the inclusion functor $\underline{P}(kG) \to \underline{M}(kG)$ induces an isomorphism K-groups tensored with Z(1/p), i.e . (see [8],[11]). This result leads to some interesting computations e.g. that for a finite group G, is a finite p-group (see [14]).

- [11] A.O. Kuku (1984): Equivariant K-theory and the cohomology of profinite groups. Lecture Notes in Mathematics 1046, Springer Verlag, Berlin 235-244

finiteness results on Higher K-theory

- Furthermore he has studied and obtained several finiteness results on Higher K-theory of modules over ‘EI’ categories, i.e. categories in which every endomorphism is an isomorphism. The theory of modules over ‘EI’ categories is a generalisation of modules over group-rings and has topological applications in the study of transformation groups since certain topological invariants reside in the K-theory groups (see [20]).

cohomology theory in form of a profinite Higher K-theory

- Let *C* be an exact category, l a rational prime. He has developed an extraordinary cohomology theory in form of a profinite Higher K-theory yielding remarkable l-completeness theorems for various exact categories *C* and in particular for the profinite higher K and G-theories , where A is any R-order in a semi simple F-algebra over number fields and p-adic fields.. This study was inspired by continuous cohomology theories rooted in algebraic topology and Arithmetic Algebraic Geometry. The results proved apply if A = RG, if R is the ring of integers in a number field or p-adic field F, he has also defined and studied continuous K-theory of p-adic orders A, and obtained several results on this construction including the fact that the even dimensional continuous higher K-groups are pro-p-groups(see [27]).

- 27 A.O. Kuku (2001): Profinite and continuous higher K-theory of exact categories, orders and group – rings. **K-Theory Journal** 22, 367-392 (2001)

Farrell-Jones Conjecture

- In a joint work with G. Tang, he has obtained interesting results on higher K-theory of virtually infinite cyclic groups V for the two types of V. When V is the semi-direct product of G and T with respect to an automorphism a of G given by inner automorphism by elements of T, they prove among other results that if R is the ring of integers in a number field F, then for all non-negative integers n, G_n(RV) is a finitely generated Abelian group and NK_n(RV) is |G|-torsion. For the second type where V is the amalgamated product of finite groups G_0 and G_1 with respect to a finite subgroup H where the index of H in G_0 and G_1 is 2 , They prove that the Nil groups of V are |H|-torsion (see [32]).

- 32 A.O. Kuku and G. Tang (2003): Higher K-theory of group rings of virtually infinite cyclic groups. **Mathematisches Annalen,** 325, 711-726 (2003).

explicit computation of the "bar" homology groups

- In a joint work with G. Tang, they obtained explicit computation of the "bar" homology groups of a non-unitial ring - a problem arising in higher K-theory and algebraic topology, see [30].

- [30] A.O. Kuku and W. Tang (2003): An explicit computation of the “Bar” homology groups of a non-unital ring. Beitrage zur algebra und Geometrie Vol 4, no 2 375-382 (2003)

algebraic structure of subgroups of the group of units of a non-commutative local ring

- He has in a joint work with M. Mahdavi-Hezavehi investigated and obtained interesting results on the algebraic structure of subgroups of the group of units of a non-commutative local ring (see [33]).

- 33 A.O. Kuku and M. Mahdavi-Hezavehi (2004): Subgroups of GL1(R) for local rings R **Communications in Algebra**", vol 32, no. 5, 1895-1902 (2004).

Non-commutative geometry

- He has also been working on Non-commutative geometry especially entire/periodic cyclic homology and K-theory of involutive Banach algebras, C*-algebras, group C*-algebras, Hopf algebras and quantum groups and studying connections (Chern characters) between K-theory and cyclic homology of these structures. More precisely, he has , in a joint work with D.N Diep and N.Q. Tho, constructed and studied non-commutative Chern characters from K-theory of compact Lie group C*-algebras and compact quantum groups to their entire/periodic cyclic homology, and proved interesting results - that the Chern characters are isomorphisms modulo torsion in the case of compact Lie group C*-algebras and compact quantum groups, (see [22],[25]).

- 22 D.N. Diep, A.O . Kuku and N.Q. Tho (1999): Non-Commutative Chern character of compact Lie group C*-algebra. **K-Theory Journal** (1999) 17(2) 195-208.
- 25 D.N. Diep, A O. Kuku and N.Q. Tho (2000): Non-Commutative Chern Characters of compact quantum groups. **Journal of Algebra** 226, 311-331 (2000).

- He has also, in a joint work with D.N. Diep, obtained some interesting results on non-commutative Chern characters of some non-compact quantum algebras see [31]. More precisely, they proved that the periodic cyclic homology groups of the quantised algebra of functions on coadjoint orbits of connected and simply connected Lie groups are isomorphic to the periodic cyclic homology of the quantised algebra of functions on coadjoint orbits of compact maximal subgroups, without localisation. They also compute the K-groups, periodic cyclic homology and Chern characters of such algebras for quantum half planes and quantum punctured complex plane.

- 31. D.N. Diep and A.O. Kuku (2001): Non-commutative Chern-Connes characters for Some non-compact quantum algebras (Preprint)

Baum-Connes conjecture

- He has also been working on quantum group theoretic formulation of the Baum-Connes conjecture - a celebrated problem in non-commutative geometry. More precisely let A be a discrete quantum group acting on a C*-algebra B and satisfying some regularity assumptions (resembling the proper G-compact action for a classical discrete group G on some space). He has in a joint work with D. Goswami (see [34]) constructed an analytic assembly map) from the A-equivariant K-homology groups to the K-theory groups.. In [36], they have provided a complete formulation of Baum-Connes conjecture for the action of discrete quantum groups as well as verified our formulation for general finite dimensional discrete quantum groups and proved surjectivity of their assembly map for the dual of SUq(2).

- 34 D. Goswami and A.O. Kuku (2002): Towards the Baum-Connes analytical assembly Map for the action of discrete quantum groups
- 36 D. Goswami and A.O. Kuku (2003): A complete formulation of Baum-Connes conjecture for the action of discrete quantum groups. '**K-theory' Journal** 30, 344-363 (2003)

wild kernels for higher K-theory of division algebras

- In a joint work with X. Guo, they have defined and studied wild kernels for higher K-theory of division algebras D over number fields. They proved among other results that it is finite. They also obtained interesting connections between the wild kernels and the subgroup of divisible elements of .of K-tgroups of the division algebras. (see [37])

- 37 X. Guo and A.O. Kuku (2006): Wild kernels for higher K-theory of division and semi-simple algebras. "**Beitrage zur Algebra and Geometrie**" / "**Contributions to Algebra and Geometry**". 47, (1) 1 - 14

K_2 of symbols

- In a joint work with Guo and Qin (see [38], they have proved that if *F* is a number field and D a finite dimensional central division F-algebra with square free index, then *K2D* is generated by Steinberg symbols *{a,b}* with *a in F*, b in D**, whereas if *F* is a global field, then for any integer n>3, there is an extension field *E* over *F* of degree *n* such that *K2E* is not generated by the Steinberg symbols *{a,b}, a in F*, b in E**.

- 38 X. Guo, A.O. Kuku and H. Qin (2003): On K2 of division algebras. **Communications in Algebra**, 33 (2005) No. 4, 1073-1081

Higher class group

- In a joint work with X. Guo, they proved that if *A* is a quaternion algebra and B an Eichler order in A, then the only p-torsion possible in even dimensional higher class groups of B are for those rational prime p which lie under prime ideals of O_F at which A is not maximal. A similar result is obtained for hereditary orders in semi-simple algebras. (See [40])

- 40 X. Guo and A.O. Kuku (2005): Higher class groups of generalized Eichler orders. Communications in algebra, 33, (2005) No 3, 709-718

equivariant higher Algebraic K-Theory

- In [42], they have constructed absolute and relative equivariant higher Algebraic K-Theory for Waldhausen categories as a generalization of the constructions in [8] for exact categories. Applications to Thomason's complicial Waldhausen categories are given as well as some finiteness results for some Waldhausen K-groups.

- 42 A.O. Kuku (2006): Equivariant Higher algebraic K-Theory for Waldhausen Categories. **“Beitrage zur Algebra und Geometrie”**”Vol 47, No.2., 583-601 (2006) .

- In [43], he proved that if R is the ring of integers in a number field F and A is any R order in a semi-simple F algebra, then K_{2n} A, G_{2n} A are finite groups and that when G is a finite p-group, SK_{2n-1} (Z G), SK_{2n-1} (Z_pG), are finite p-groups.

- 43 A. O. Kuku (2005): Finiteness of Higher K-groups of orders and grouprings. "**K-theory**" Journal., 36, 51 -58 (2005)

- Let G be an algebraic group over a field F.In [45] , he studied and computed equivariant higher K-groups as well as profinite equivariant higher K-groups for some G-schemes over number fields and p-adic fields. He also obtained among others, some finiteness and l-completeness results for twisted flag varieties etc.

- 45 A. O. Kuku (2011) Profinite Higher Algebraic K-theory of twisted flag varieties

  Accepted and to appear in "AFRIKA MATEMATIKA" Published by Springer.

Rational Higher K-theory and G-theory

- In [46], he proved that the rational Higher K-theory and G-theory for twisted power Series rings over arbitrary orders are isomorphic as well as isomorphic to rational K-theory of twisted power series rings over semi-simple algebras over number fields. He also proved some finiteness results for negative K-theory of such rings as well as some l-completeness and other results for profinite K-theory of such rings.

- 46 A. O. Kuku (2008) Higher Algebraic K-theory for twisted Laurent series rings over orders and semi-simple algebras. "ALGEBRAS AND REPRESENTATION THEORY" (2008) 11: 355-368

cohomology theory

- In [47], he constructed a cohomology theory in the category of generalized Bredon co-efficient systems in a purely categorical setting in order to generalize classical Bredon cohomology theory and show that this theory constructed in general categorical terms indeed yields Bredon cohomolgy for finite, discrete, and profinite groups. They also study Higher K-theory for the category of finitely generated (resp. finitely generated projective) objects in the category of generalized Bredon co-efficient systems and obtain some finiteness results.

- 47 A. O. Kuku (2009) Cohomology and Higher K-theory for generalized Bredon co-efficient systems. (Preprint)

higher K-theory of p-adic orders

- In [48], he studies higher K-theory of p-adic orders, and twisted polynomial and power series rings over p-adic orders. For higher K-theory of p-adic orders, he obtains a partial solution to an open question, and in the three cases, obtain some p-torsion results. He also prove that higher K- and G-theories of twisted Laurent series rings over p-adic orders are rationally isomorphic.

- 48 A. O. Kuku (2011) Higher Algebraic K-theory of p-adic orders and twisted Polynomial and Laurent series rings over p-adic orders. (accepted and to appear in "COMMUNICATIONS IN ALGEBRA")

profinite higher K-groups

- [49] is focussed on studying equivariant exact categories for the actions of finite and algebraic groups as well as computing their higher and profinite higher K-groups. For algebraic groups , several results are presented for twisted flag varieties and Severi-Brauer varieties.

- 49 A. O. Kuku (2010) Higher Algebraic K-theory of G-representations for the actions of finite and algebraic groups. In Group Theory— Classes, Representations and Connections and Applications. NOVA SCIENCE PUBLISHERS: pp 41-82.(2010)

Introduction to K-theory and Index Theory

- [50] is the essential contents of the series of lectures he gave titled “Introduction to K-theory and Index Theory.”at the International “CIMPA/UNESCO/BURKINAFASO Workshop on ‘Index Theory and Interractions with Physics”. The lecture was aimed at introducing the participants to Index theory via K-theoretic methods.

- 50 A. O. Kuku (2010) Introduction to K-theory and Index theory: Lectures given at the Intenational CIMPA/UNESCO/BURKINA-FASO workshop on”Index theory and Interractions with Physics” .at the University Ouagadugou, Burkina Faso, May 2009. On line publication of CIMPA—Centre International de Mathematiques Pures et Appliquees.(2010)

Abstract Algebra
Group Representation and its Applications

- Professor Kuku has published a book titled 'Abstract Algebra' xvii + 419 pages suitable for the use of honors undergraduates and beginning graduate students in mathematics (see [51]). Together with J Rawnsley and E. Thoma. He also published a book "Group Representation and its Applications (see [52]).

- 51 A. O. Kuku (2010) Trends in Contemporary Mathematics: Illustrations from K-theory. In Proceedings of the Workshop on”Capacity building For Mathematical Sciences Lecturers in Tertiary Institutions. National Mathematical Centre, Abuja, Nigeria.

RepresentationTheory and Higher Algebraic K-theory

- A. O. Kuku (Book) (2007) RepresentationTheory and Higher Algebraic K-theory, CHAPMAN AND HALL (Taylor Francis Group) . xxviii + 442 pages